UNRESTRAINED BEAMS

SUMMARY:

- Beams bent about the major axis may fail by buckling in a more flexible plane.
- This form of buckling involves both lateral deflection and twisting - *lateral-torsional buckling.*
- A design approach for beams prone to failure by lateral-torsional buckling must account for a large number of factors - including section shape, the degree of *lateral restraint*, type of loading, *residual stress pattern* and *initial imperfections.*
- Stocky beams are unaffected by lateral torsional buckling and capacity is governed by the plastic resistance moment of the cross section.
- Slender beams have capacities close to the theoretical elastic critical moment.
- Many practical beams are significantly adversely affected by inelasticity and geometrical imperfections, hence elastic theory provides an upper band solution.
- A design expression linking the plastic capacity of stocky beams with the elastic behaviour of slender beams is provided by a *reduction factor* for lateral torsional buckling, χ_{LT}.

OBJECTIVES:

- Introduce the phenomenon of lateral torsional stability.
- Identify the controlling parameters.
- Present a simple analogy between the behaviour of the compression flange and the strut flexural buckling.
- Understand the significance of the terms in the elastic torsional buckling equations.
- Briefly explain the reasons why the elastic theory, requires modification before being used as a basis for the unrestrained beam design rules.
- Apply the EC3 rules to the design of a simply supported laterally unrestrained beam.
- Recognize practical applications where lateral torsional buckling is unlikely to present a problem.
- Briefly describes the role of bracing and how to improve its efficiency.

REFERENCES:

- Narayanan, R., ed., "Beams and Beam Columns: Stability and Strength", Applied Science Publishers, 1983
- Chen,W.F. & Atsuta,T., "Theory of Beam Columns V. 2, Space Behaviour and Design", McGraw Hill, 1977.
- Timoshenko, S.P. and Gere, J.M., "Theory of Elastic Stability", Second Edition, McGraw Hill, 1962.
- Eurocode 3 Design of steel structures Part 1.1 General rules and rules for buildings.
- Trahair, N.S. and Bradford, M.A., "The Behaviour and Design of Steel Structures", E&F Spon, 1994.

CONTENTS:

1. INTRODUCTION.

When designing a steel beam it is usual to think first of the need to provide adequate strength and stiffness against vertical bending.

This leads naturally to a cross-sectional shape in which the stiffness in the vertical plane is much greater than that in the horizontal plane.

Sections normally used as beams have the majority of their material concentrated in the flanges, which are relatively narrow so as to prevent local buckling.

The need to connect beams to adjacent members with ease normally suggests the use of an open section, for which the torsional stiffness will be comparatively low.

Figure 1, which compares section properties for four different shapes of equal area, shows that the high vertical bending stiffness of typical beam sections is obtained at the expense of both horizontal bending and torsional stiffness.

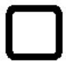
Section type	Flat	H-Sections (Typical)	I-Sections (Typical)	Hollow sections (Typical)
Section properties				
A Area	1	1	1	1
I_y (Vertical loading)	1	0,35	1	0,2
I_z (Horizontal loading)	0,2	3,5	1	3,5
J (Twisting)	1	1	1	100

Figure 1 – Types of cross section used as beams showing relative values of section properties.

It is known from our understanding of the behaviour of struts that, whenever a slender structural element is loaded in its stiff plane (axially in the case of the strut), there exists a tendency for it to fail by buckling in a more flexible plane (by deflecting sideways in the case of the strut).

In the case of beam bent about its major axis, failure may occur by a form of buckling which involves both lateral deflection and twisting - lateral-torsional buckling.

Figure 2 illustrates the phenomenon with a slender cantilever beam loaded by a vertical end load.

If the cantilever was perfectly straight and the cross section initially stress free and perfectly elastic, the tip of the cantilever would deflect only in the vertical plane with no out of plane deflection until the applied moment reached a critical value at which the beam buckles by deflecting laterally and twisting.

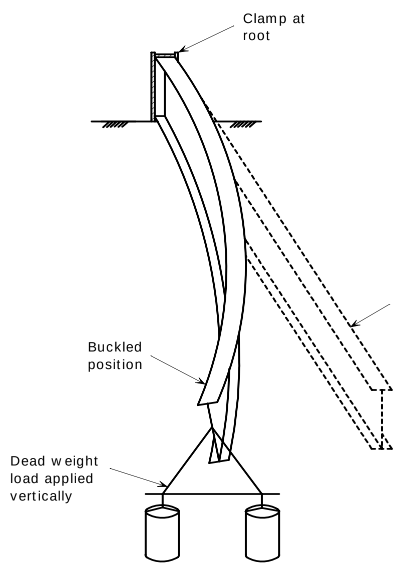


Figure 2 - Lateral-torsional buckling of a slender cantilever beam.

Although it involves both a lateral deflection (u) and twisting about a vertical axis through the web (ϕ), Figure 3, this type of instability is quite similar to the simpler flexural buckling of an axially loaded strut.

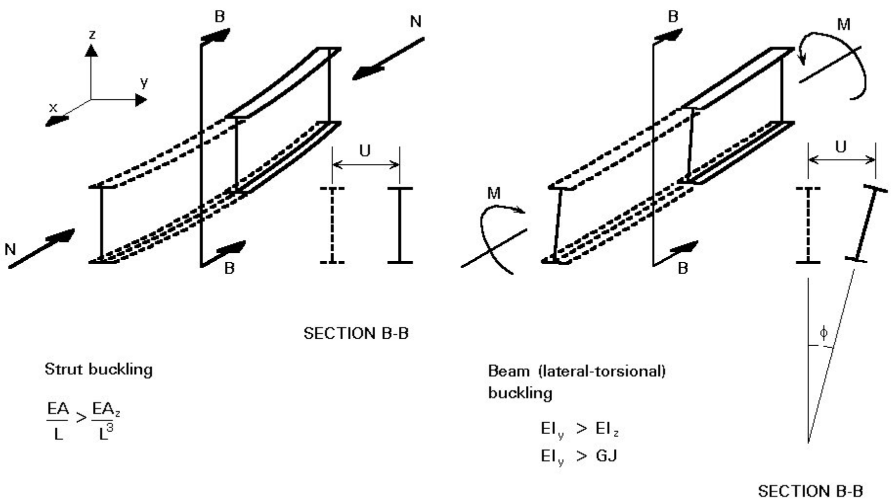


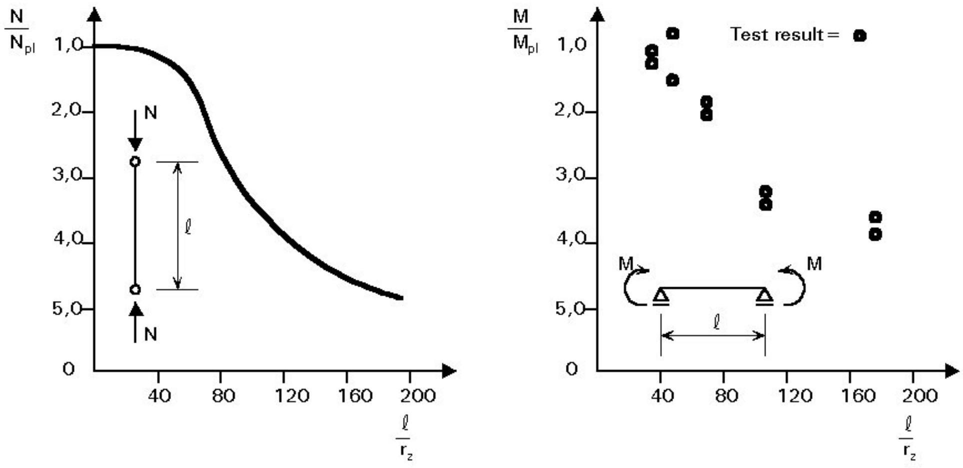
Figure 3 – Similarity between strut buckling and beam buckling.

Of course, many types of construction effectively prevent this form of buckling, thereby enabling the beam to be designed with greater efficiency as fully restrained.

It is important to realise that during erection of the structure certain beams may well receive far less lateral support than will be the case when floors, decks, bracings, etc., are present, so that stability checks, at this stage, are also necessary.

Lateral-torsional instability influences the design of laterally unrestrained beams in the same way that flexural buckling influences the design of columns.

Thus the bending strength will now be a function of the beam's slenderness, as indicated in Figure 4, requiring the use in design of an iterative procedure similar to the use of column curves in strut design.



Typical strut design curve in ultimate load format

Test data for 305 x 102 UB 28 sections as laterally unrestrained beams

Figure 4 – Dependence of the beam strength on unrestrained length and analogy with column strength.

A design approach for beams prone to failure by lateral-torsional buckling must of necessity account for a large number of factors (including section shape, the degree of lateral restraint, type of loading, residual stress pattern and initial imperfections) and is therefore relatively complex.

2. ELASTIC BUCKLING OF A SIMPLY SUPPORTED BEAM.

Figure 5 shows a perfectly elastic, initially straight I beam loaded by equal and opposite end moments about its major axis (ie. in the plane of the web).

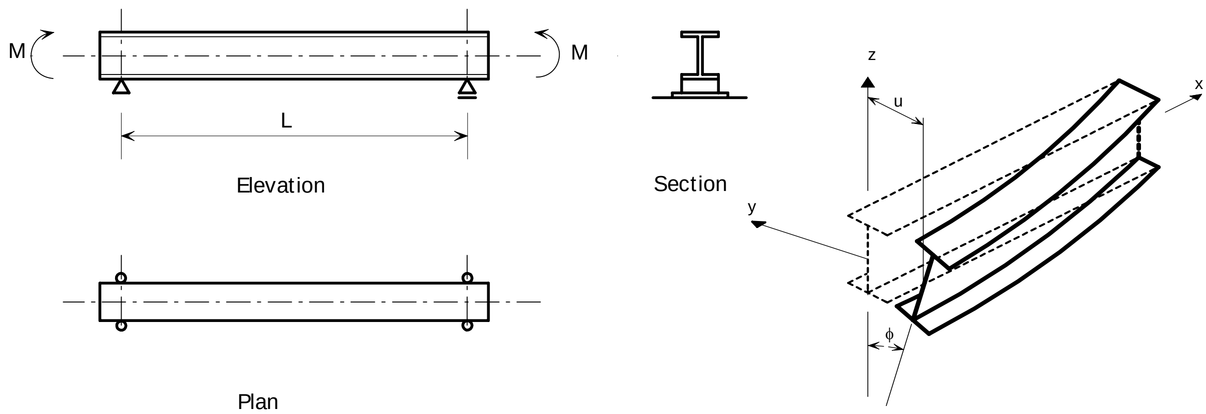


Figure 5 - Lateral torsional buckling of a simple I beam under uniform moment.

The beam is unrestrained along its length except at each end where the sections is prevented from twisting and lateral deflection but is free to rotate both in the plane of the web and on plan.

The buckled shape and resultant deformations are also shown in the figure (note only half of the beam is shown, the deformations are at the midspan).

The buckling moment can be determined by equating the disturbing effect of the applied end moments, acting through the buckling deformations, to the internal (bending and torsional) section resistance.

Derivation of Governing Equations

The deformed state of the beam, Figure 6, identifies the deflections (u and v) and the twist (ϕ).
A new co-ordinate system ξ η ζ , which deflects with the beam, is also illustrated.

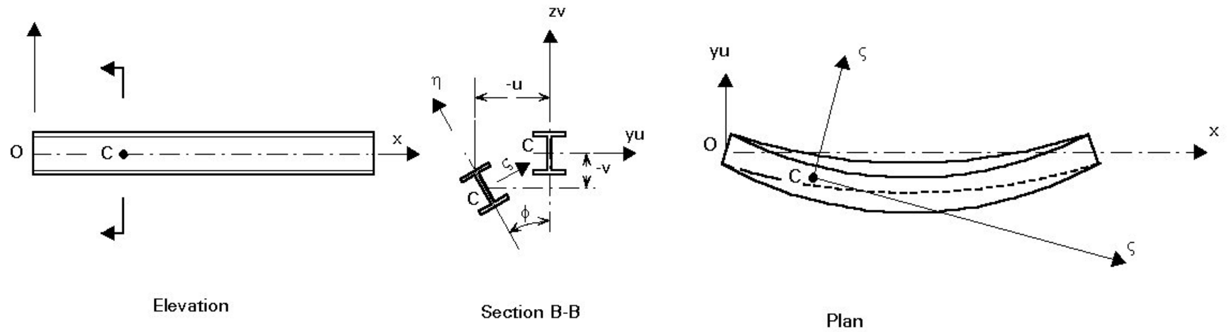


Figure 6 – Deformation of a beam.

Bending in the ξ ζ and η ζ planes and twisting about the ζ axis are governed by:

$$EI_y \frac{d^2 v}{dx^2} = M_\xi \qquad (1)$$

$$EI_z \frac{d^2 u}{dx^2} = M_\eta \qquad (2)$$

$$GI_t \frac{d\phi}{dx} - EI_w \frac{d^3\phi}{dx^3} = M_\zeta \qquad (3)$$

In Equations (1) and (2) the flexural rigidities and curvatures in the ξ ζ and the η ζ planes have been replaced by the values for the yx and zx planes, on the basis that ϕ is a small angle.

Equation (3) includes both mechanisms available in a thin-walled section to resist twist; the first term corresponds to that part of the applied torque which is resisted by the development of shear stresses, whilst the second term allows for the influence of restrained warping.

This latter phenomenon arises as a direct result of the type of axial flange deformation, illustrated in Figure 7a, that occurs in an I-section subject to equal and opposite end torques.

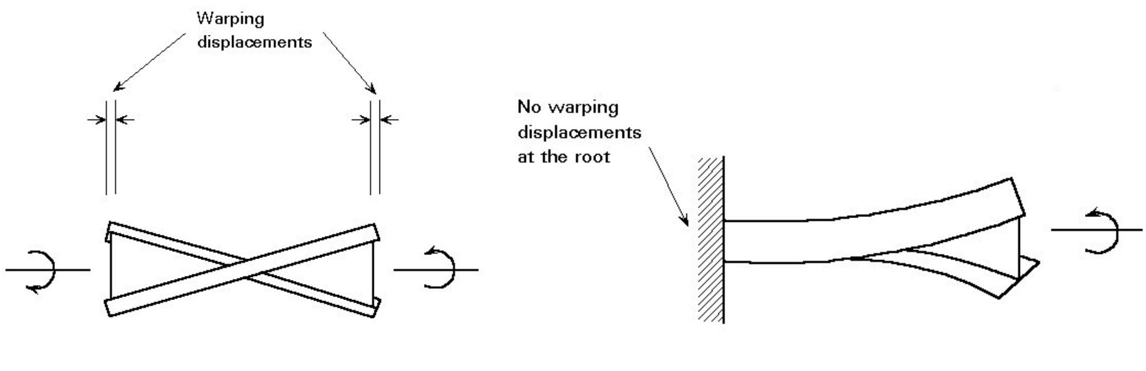


Figure 7 – Torsion boundary conditions.

The two flanges tend to bend in opposite senses about a vertical axis through the web, with the result that originally plane sections do not remain plane.

On the other hand, for the cantilever of Figure 7b, it is clear that warping deformations must be at least partly inhibited elsewhere along the span, since they cannot occur at the fixed end.
This induces additional axial stresses in the flanges; the pair of couples, or bimoment, due to this additional stress system provides part of the section's resistance to twist.

In the case of lateral instability, restraint against warping arises as a result of adjacent cross-sections wanting to warp by different amounts.

For an I-section, the relative magnitudes of the warping constant I_w and the torsion constant I_t are:

$$I_w = I_x \frac{h_f^2}{4} \quad \text{and} \quad I_t = \frac{2bt^3}{3} + \frac{dt_w^3}{3}$$

They will be affected principally by the thickness of the component plates and by the section depth.

For compact column-type sections the first term in Equation (3) will tend to provide most of the twisting resistance, whilst the second term will tend to become dominant for deeper beam shapes.

Consideration of the buckled shape using Figures 5, 6 and 8 enables the components of the applied moment in the $\xi\,\zeta$ and $\eta\,\zeta$ planes and about the ζ axis to be obtained as:

$$M_\xi = M\cos\phi \text{ , } M_\eta = M\sin\phi \text{ , } M_\zeta = M\sin\alpha \tag{4}$$

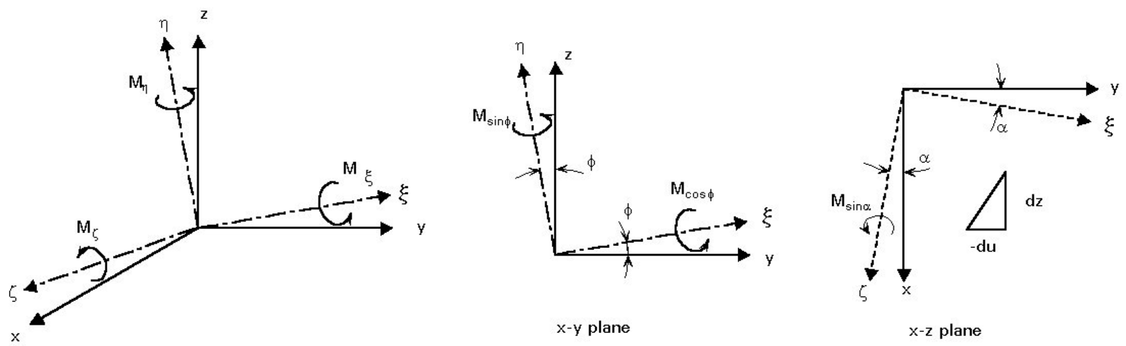


Figure 8 – Resolution of applied end moment M into components about the deformed axes.

Since ϕ is small, $\cos\phi \approx 1$ and $\sin\phi \approx \phi$, whilst Figure 8 shows that $\sin\alpha$ may be approximated by $-\frac{du}{dx}$. Thus Equations (1) - (3) may be written as:

$$EI_y \frac{d^2v}{dx^2} = M \tag{5}$$

$$EI_z \frac{d^2u}{dx^2} = M\phi \tag{6}$$

$$GI_t \frac{d\phi}{dx} - EI_w \frac{d^3\phi}{dx^3} = M\frac{du}{dx} \tag{7}$$

Since Equation (5) contains only the vertical deflection (v), it is independent of the other two; it controls the in-plane response of the beam.

Equations (6) and (7) are coupled in u and ϕ, the buckling deformations; their solution gives the value of elastic critical moment (M_{cr}) at which the beam becomes unstable. Combining them gives:

$$EI_w \frac{d^4\phi}{dx^4} - GI_t \frac{d^2\phi}{dx^2} + \frac{M^2}{EI_z}\phi = 0 \tag{8}$$

The solution of Equation (8) is made far simpler if the warping stiffness (I_w) is assumed to be zero.

The results obtained are then directly applicable to beams of narrow rectangular cross-section but are conservative for the normal range of I-sections. Equation (8) therefore reduces to:

$$\frac{d^2\phi}{dx^2} + \frac{M^2}{EI_z GI_t \phi} = 0 \tag{9}$$

Putting $\mu^2 = \frac{M^2}{EI_z GI_t}$ enables the solution to be written as:

$$\phi = A\cos \mu x = B\sin \mu x \tag{10}$$

Noting the boundary conditions at both ends gives

$$\text{When } x = 0,\ \phi = 0;\ \text{then } A = 0 \tag{11}$$

When $x = L$, $\phi = 0$; then $B\sin \mu L = 0$

$$\text{and either } B = 0, \text{ or} \tag{12}$$
$$\sin \mu L = 0 \tag{13}$$

The first possibility gives the unbuckled position whereas the second gives:

$$\mu L = 0,\ \pi,\ 2\pi \tag{14}$$

and the first non-trivial solution is:

$$\mu L = \pi \tag{15}$$

which gives:

$$M_{cr} = \left(\pi/L\right)\sqrt{EI_z GI_t} \tag{16}$$

Note that the form of Equation (9) is identical to the form of the basic Euler strut equation.

Returning to the original Equation (8), this may be solved to give:

$$M_{cr} = \left(\pi/L\right)\sqrt{EI_z GI_t}\sqrt{1 + \left(\frac{\pi^2 EI_w}{L^2 GI_t}\right)} \tag{17}$$

The inclusion of warping effects enhances the value of M_{cr} depending on the relative values of EI_w & GI_t.

The presence of the flexural (EI_z) and torsional (GI_t and EI_w) stiffnesses of the member in the equation is a direct consequence of the lateral and torsional components of the buckling deformations.

The relative importance of these items will be a reflection of the type of cross section considered. Length is also important, entering both directly and indirectly via the $\pi^2 EI_w/L^2 GI_t$ term.

It is not possible to simplify Equation (17) by omitting terms without imposing limits on the range of application of the resulting approximate solution. Another way of presenting equation 17 is:

$$M_{cr} = \frac{\pi^2 EI_z}{L^2}\left[\frac{I_w}{I_z} + \frac{L^2 GI_t}{\pi^2 EI_z}\right]^{0,5} \tag{18}$$

Figure 9 illustrates this point by comparing the elastic critical moment of a box section (which has high flexural and torsional stiffness) with open sections of various shapes.

The region of the curves for both I-sections of low length/depth ratios corresponds to the situation in which the second square root term value in Equation (17) adopts a value significantly in excess of unity.

Since warping effects will be very important for thin plates deep sections, it follows that the $\pi^2 EI_w/L^2 GI_t$ term will, in general, tend to be large for short deep girders and small for long shallow beams.

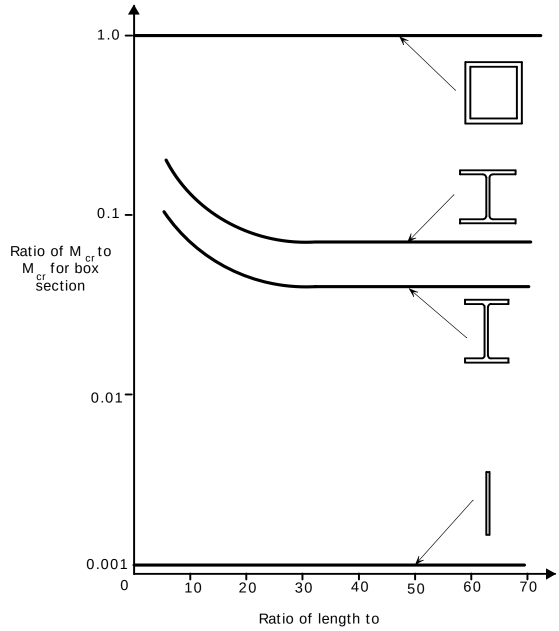
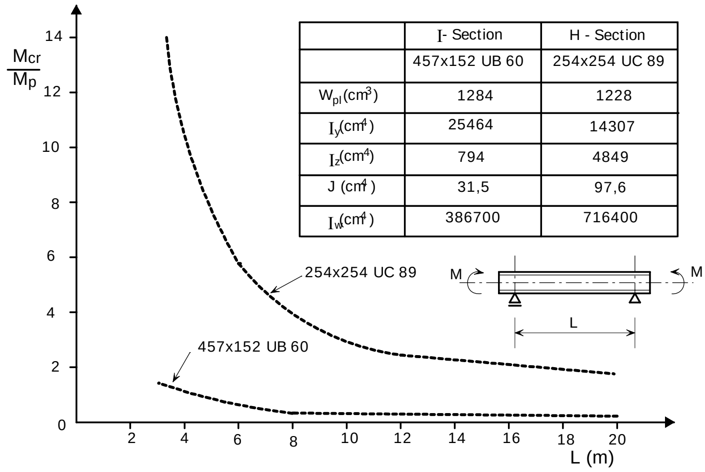


Figure 9 - Effect of cross section shape on theoretical elastic critical moment.

Figure 10 compares values of the elastic critical moment (M_{cr}) for an I beam and a column section with similar in plane plastic moment capacities.



	I- Section	H - Section
	457x152 UB 60	254x254 UC 89
W_{pl} (cm^3)	1284	1228
I_y (cm^4)	25464	14307
I_z (cm^4)	794	4849
J (cm^4)	31,5	97,6
I_w (cm^6)	386700	716400

I

Figure 10 - Comparison of elastic critical moments for I and H sections.

Lateral-torsional buckling is a potentially more significant design consideration for a beam section that is much less stiff laterally and torsionally.

3. SIMPLE PHYSICAL MODEL.

Before considering the analysis of the problem, it is useful to attempt to gain an insight into the physical behaviour by considering a simplified model.

Since bending of an I-section beam is resisted principally by the tensile and compressive forces developed in two flanges, as shown in Figure 11, the compression flange may be regarded as a strut.

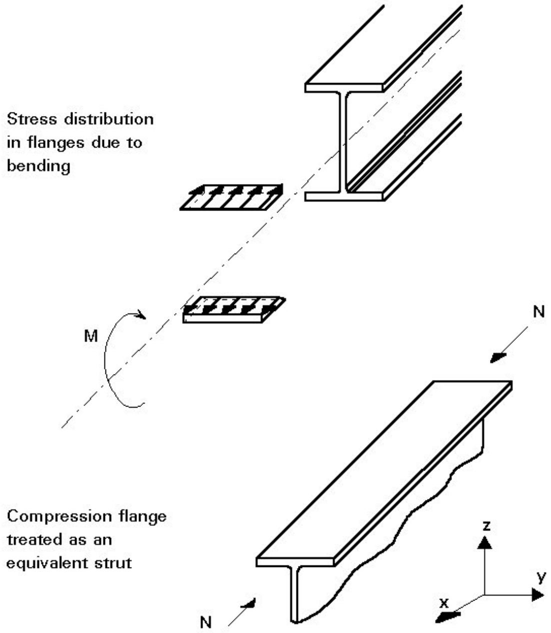


Figure 11 – Approximation of beam buckling problem as a strut problem.

Compression members generally buckles in the weaker direction i.e. the flange buckles downwards.

However, this is prevented by the presence of the web.

Therefore the flange is forced to buckle sideways, which will induce some degree of twisting in the section as the web too is required to deform.

Whilst this approach neglects the real influence of torsion and the role of the tension flange, it does approximate the behaviour of very deep girders with very thin webs or of trusses or open web joists.

Indeed, early attempts at analysing lateral-torsional buckling started with this approach.

The compression flange/strut analogy is also helpful in understanding:

1. The buckling load of the beam depends on its unbraced span, i.e. the distance between points at which lateral deflection is prevented, and on its lateral bending stiffness (EL_z) because strut resistance $\propto EL_z/L^2$.
2. The shape of the cross-section may be expected to have some influence, with the web and the tension flange being more important for relatively shallow sections, than for deep slender sections. In the former case the proximity of the stable tension flange to the unstable compression flange increases stability and also produces a greater twisting of the cross-section. Thus torsional behaviour becomes more important.
3. For beams under non-uniform moment, the force in the compression flange will no longer be constant, as shown in Figure 12. Therefore such members might reasonably be expected to be more stable than similar members under a more uniform pattern of moment.
4. End restraint which inhibits development of the buckled shape, shown in Figure 3, increases the beam stability. Consideration of the buckling deformations (u and ϕ) should make it clear that this refers to rotational restraint in plan, i.e.about the z-axis (refer back to Figure 11 and 3). Rotational restraint in the vertical plane affects the pattern of moments in the beam (and may also lead to increased stability) but does not directly alter the buckled shape, Figure 13.

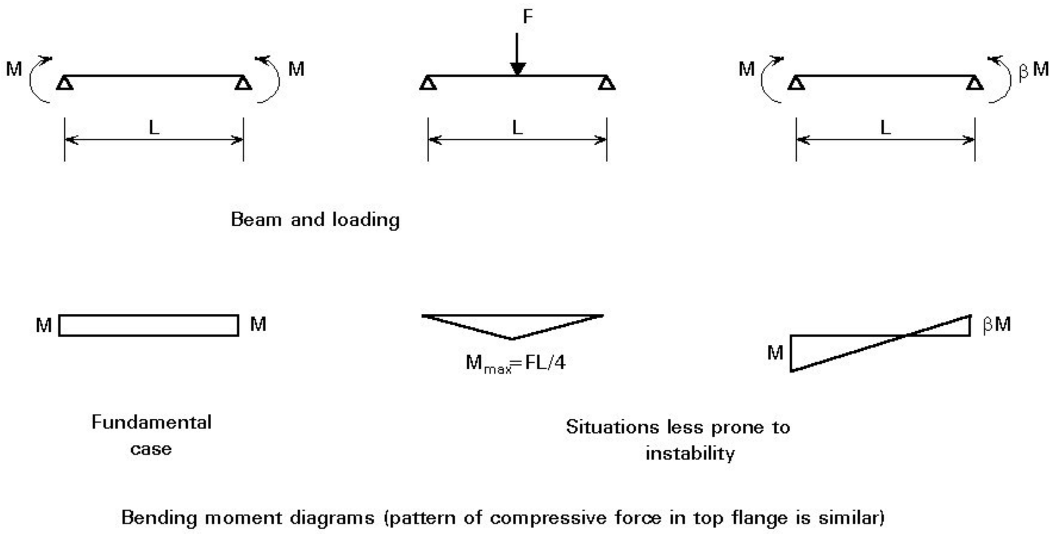


Figure 12 – effect on non-uniform moment on lateral –torsional buckling.

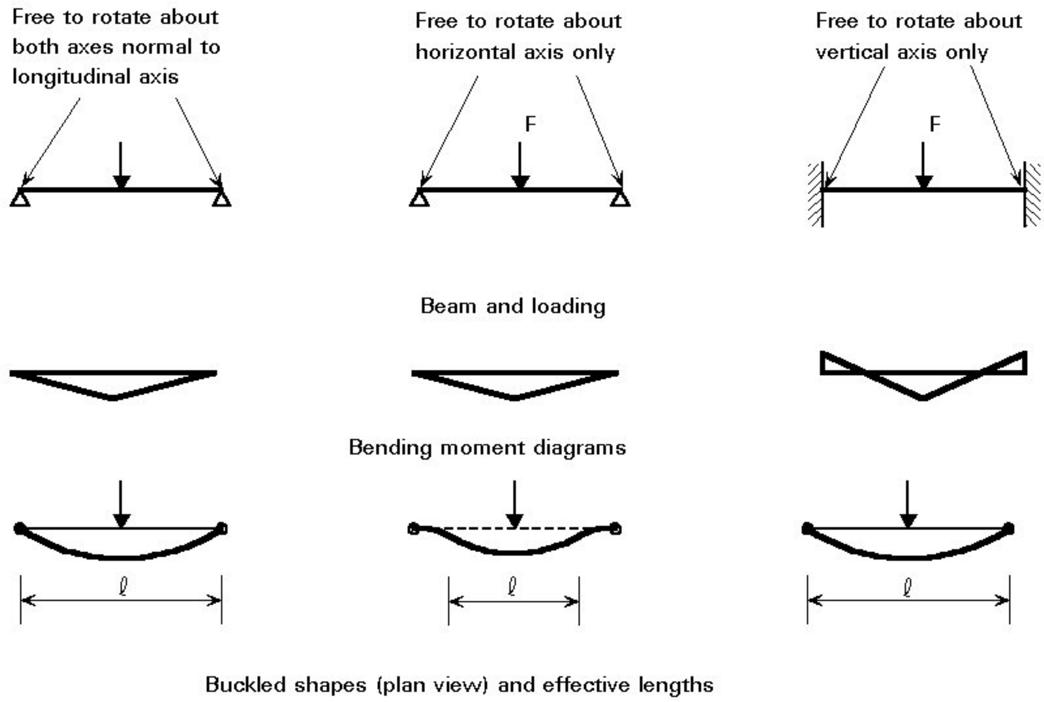


Figure 13 – Effect of restraint in plane or elevation on lateral –torsional buckling.

4. BRACING AS A MEANS OF IMPROVING PERFORMANCE.

Bracing may be used to improve the strength of a beam that is liable to lateral-torsional instability.

Two requirements are necessary:

1. The bracing must be sufficiently stiff to hold the braced point effectively against lateral movement (this can normally be achieved without difficulty).
2. The bracing must be sufficiently strong to withstand the forces transmitted to it by the main member (these forces are normally a % of the force in the braced member compression flange).

Providing these two conditions are satisfied, then the full in-plane strength of a beam may be developed through braces at sufficiently close spacing.

Figure 14, illustrates buckled shapes for beams with intermediate braces, and shows how this buckling involves the whole beam.

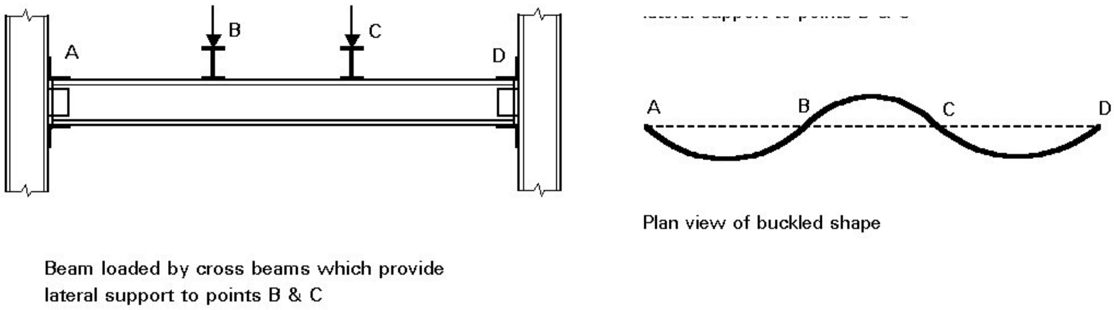


Figure 14 – Buckling of beams provided with lateral bracing.

In theory, bracing should prevent either lateral or torsional displacement from occurring.

In practice, consideration of the buckled shape of the beam cross-section shown in Figure 3 suggests that bracing is potentially most effective when used to resist the largest components of deformation, i.e. a lateral brace attached to the top flange is likely to be more effective than another on the bottom flange.

5. DEVELOPMENT OF A DESIGN APPROACH.

Real beams are not perfectly straight nor is the material elastic.

Figure 15 shows the effects of residual stresses and strain hardening on the lateral buckling strength.

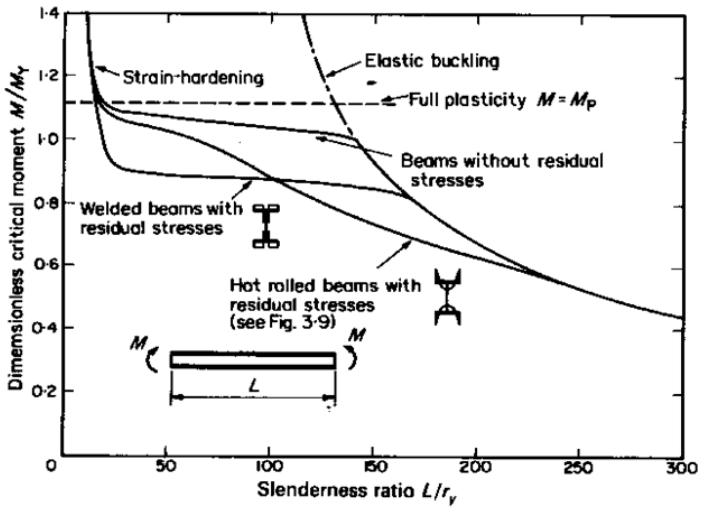


Figure 15 - Lateral buckling strengths of simply supported I beams.

Note that at high slenderness values the behaviour is well represented by elastic buckling theory but for stocky beams there is a complex interplay as inelastic behaviour causes a reduction in capacity, and for very stocky beams the capacity is limited by the plastic resistance of the section.

Application of a theoretical treatment of the problem would be too complex for routine design so a combination of theory and test results is required to produce a reliable (safe) design approach.

Figure 16 compares a typical set of lateral torsional buckling test data with the theoretical elastic critical moments given by eqn 1.

A non-dimensionalised form of plot has been used which permits results from different test series (which have different cross-sections and material strengths) to be directly compared via a non-dimensional slenderness, $\bar{\lambda}_{LT}$.

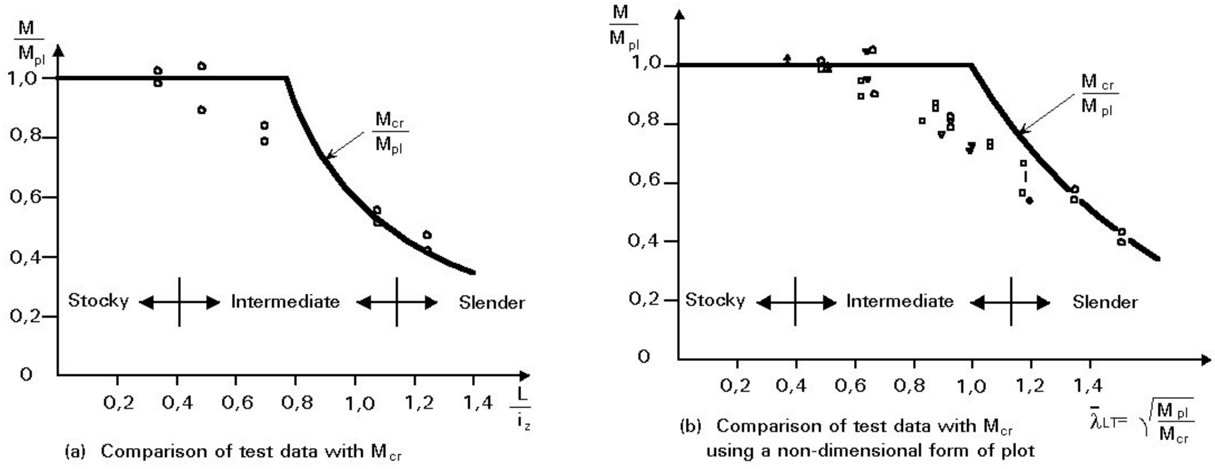


Figure 16 - Comparison of test data with theoretical elastic critical moments

For stocky beams ($\bar{\lambda}_{LT} \langle 0.4$) the capacity is unaffected by lateral torsional buckling and is governed by the plastic resistance moment of the cross section.

Slender beams ($\bar{\lambda}_{LT} \langle 1.2$) have capacities close to the theoretical elastic critical moment, M_{cr}.

Only in the case of beams in region 1 does lateral stability not influence design; such beams can be designed using the already described methods.

For beams in region 2, which covers much of the practical range of beams without lateral restraint, design must be based on considerations of inelastic buckling suitably modified to allow for geometrical imperfections, residual stresses, etc.

Thus both theory and tests must play a part, with the inherent complexity of the problem being such that the final design rules are likely to involve some degree of empiricism.

It should be noted that sections of the type illustrated in Figure 17b, with one axis of symmetry, e.g.channels, may only be included if the section is bent about the axis of symmetry, i.e. loads are applied through the shear centre parallel to the web of the channel.

Singly-symmetrical sections bent in the other plane, i.e. an unequal flanged I-section bent about its major-axis as shown in Figure 174c, may only be treated by an extended version of the theory, principally because the section's shear centre no longer lies on the neutral axis.

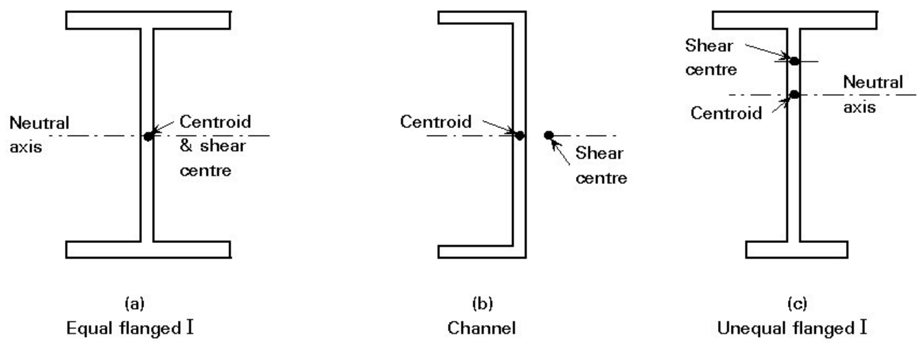


Figure 17- - Equal flanged section and examples of sections with one axis of symmetry.

A design expression linking the plastic capacity of stocky beams with the elastic behaviour of slender beams is required. EC3 achieves this by use of a reduction factor for lateral torsional buckling, χ_{LT}.
The design buckling resistance moment ($M_{b.Rd}$) of a laterally unrestrained beam is thus taken as:

$$M_{b.Rd} = \chi_{LT}\, \beta_w\, W_{pl.y}\, f_y/\gamma_{m1} \qquad (19)$$

Eurocode 3 5.5.2 (1) (5.48) or eq. 6.55

which is effectively the plastic resistance of the section multiplied by the reduction factor (χ_{LT}).

Figure 18 shows the relationship between χ_{LT} and the non-dimensional slenderness, $\bar{\lambda}_{LT}$.

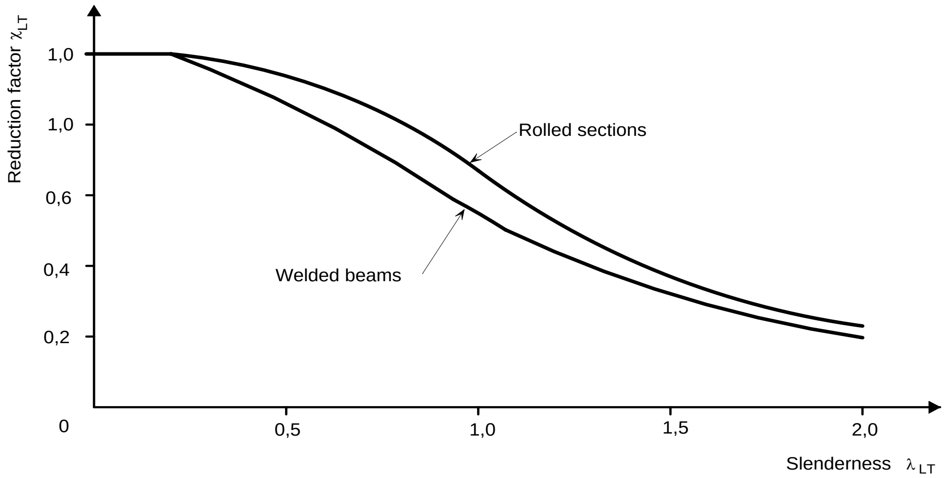


Figure 18 Lateral-torsional buckling reduction factor

The curves shown are expressed by

$$\chi_{LT} = \frac{1}{\phi_{LT} + \left[\phi_{LT}^{\,2} - \bar{\lambda}_{LT}^{\,2}\right]^{0,5}} \qquad (20)$$

Eurocode 3 5.5.2 (2) (5.49) or eq. 6.56

where :

$$\phi_{LT} = 0,5\left[1 + a_{LT}(\bar{\lambda}_{LT} - 0.2)) + \bar{\lambda}_{LT}^{\,2}\right] \qquad (21)$$

a_{LT} is an imperfection factor, 0,21 for rolled sections and 0,49 for welded sections, with their more severe residual stresses.

$\bar{\lambda}_{LT}$, the non-dimensional slenderness, defined as $\sqrt{M_{plRd}/M_{cr}}$, may be calculated either by calculating the plastic resistance moment and elastic critical moment by the relationship:

$$\bar{\lambda}_{LT} = \left[\frac{\lambda_{LT}}{\lambda_1}\right]\beta_w^{\,0,5} \qquad \text{or} \qquad \bar{\lambda}_{LT} = \sqrt{\frac{W_y f_y}{M_{cr}}} \qquad (22)$$

Eurocode 3 5.5.2 (5) or eq. 6.56

Where:

$$\lambda_1 = \pi\sqrt{\frac{E}{fy}} \qquad (23)$$

and λ_{Lt} may be calculated with appropriate equations for a variety of section shapes (EC 3 App. F.2.2.).

For example, for any plain I or H section with equal flanges, and subject to uniform moment with simple end restraints,

$$\lambda_{LT} = \frac{L/i_z}{\left[1+\frac{1}{20}\left[\frac{L/i_z}{h/t_f}\right]^2\right]^{0,25}} \quad (24)$$

Eurocode 3
F.2.2 (5)
(F.21)

This expression is a conservative approximation for any uniform plain I or H shape with equal flanges (Annex F2 of EC3).

6.3.2 Uniform members in bending

6.3.2.1 Buckling resistance

(1) A laterally unrestrained beam subject to major axis bending shall be verified against lateral-torsional buckling as follows:

$$\frac{M_{Ed}}{M_{b,Rd}} \leq 1,0 \qquad (6.54)$$

where M_{Ed} is the design value of the moment

$M_{b,Rd}$ is the design buckling resistance moment.

(2) Beams with sufficient restraint to the compression flange are not susceptible to lateral-torsional buckling. In addition, beams with certain types of cross-sections, such as square or circular hollow sections, fabricated circular tubes or square box sections are not susceptible to lateral-torsional buckling.

(3) The design buckling resistance moment of a laterally unrestrained beam should be taken as:

$$M_{b,Rd} = \chi_{LT} W_y \frac{f_y}{\gamma_{M1}} \qquad (6.55)$$

where W_y is the appropriate section modulus as follows:

- $W_y = W_{pl,y}$ for Class 1 or 2 cross-sections
- $W_y = W_{el,y}$ for Class 3 cross-sections
- $W_y = W_{eff,y}$ for Class 4 cross-sections

χ_{LT} is the reduction factor for lateral-torsional buckling.

NOTE 1 For determining the buckling resistance of beams with tapered sections second order analysis according to 5.3.4(3) may be performed. For out-of-plane buckling see also 6.3.4.

NOTE 2B For buckling of components of building structures see also Annex BB.

(4) In determining W_y holes for fasteners at the beam end need not to be taken into account.

6.3.2.2 Lateral torsional buckling curves – General case

(1) Unless otherwise specified, see 6.3.2.3, for bending members of constant cross-section, the value of χ_{LT} for the appropriate non-dimensional slenderness $\bar{\lambda}_{LT}$, should be determined from:

$$\chi_{LT} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \bar{\lambda}_{LT}^2}} \quad \text{but } \chi_{LT} \leq 1,0 \qquad (6.56)$$

where $\Phi_{LT} = 0,5\left[1 + \alpha_{LT}\left(\bar{\lambda}_{LT} - 0,2\right) + \bar{\lambda}_{LT}^2\right]$

α_{LT} is an imperfection factor

$$\bar{\lambda}_{LT} = \sqrt{\frac{W_y f_y}{M_{cr}}}$$

M_{cr} is the elastic critical moment for lateral-torsional buckling

(2) M_{cr} is based on gross cross sectional properties and takes into account the loading conditions, the real moment distribution and the lateral restraints.

NOTE The imperfection factor α_{LT} corresponding to the appropriate buckling curve may be obtained from the National Annex. The recommended values α_{LT} are given in Table 6.3.

Table 6.3: Imperfection factors for lateral torsional buckling curves

Buckling curve	a	b	c	d
Imperfection factor α_{LT}	0,21	0,34	0,49	0,76

The recommendations for buckling curves are given in Table 6.4.

Table 6.4: Lateral torsional buckling curve for cross sections using equation (6.56)

Cross-section	Limits	Buckling curve
Rolled I-sections	$h/b \le 2$ $h/b > 2$	**a** **b**
Welded I-sections	$h/b \le 2$ $h/b > 2$	**c** **d**
Other cross-sections	-	**d**

(3) Values of the reduction factor χ_{LT} for the appropriate non-dimensional slenderness $\bar{\lambda}_{LT}$ may be obtained from Figure 6.4.

(4) For slendernesses $\bar{\lambda}_{LT} \le 0{,}2$ (or $\bar{\lambda}_{LT} \le 0{,}4$ (see 6.3.2.3)) or for $\frac{M_{Ed}}{M_{cr}} \le 0{,}04$ (or $\frac{M_{Ed}}{M_{cr}} \le 0{,}16$ (see 6.3.2.3)) lateral torsional buckling effects may be ignored and only cross sectional checks apply.

6.3.2.3 Lateral torsional buckling curves for rolled sections or equivalent welded sections

(1) For rolled or equivalent welded sections in bending the values of χ_{LT} for the appropriate non-dimensional slenderness may be determined from

$$\chi_{LT} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \beta\bar{\lambda}_{LT}^2}} \quad \text{but} \quad \begin{cases} \chi_{LT} \le 1{,}0 \\ \chi_{LT} \le \dfrac{1}{\bar{\lambda}_{LT}^2} \end{cases} \qquad (6.57)$$

$$\Phi_{LT} = 0{,}5\left[1 + \alpha_{LT}\left(\bar{\lambda}_{LT} - \bar{\lambda}_{LT,0}\right) + \beta\bar{\lambda}_{LT}^2\right]$$

NOTE The parameters $\bar{\lambda}_{LT,0}$ and β and any limitation of validity concerning the beam depth or h/b ratio may be given in the National Annex. The following values are recommended for rolled sections:

$\bar{\lambda}_{LT,0} = 0{,}4$ (maximum value)

$\beta = 0{,}75$ (minimum value)

The recommendations for buckling curves are given in Table 6.5.

Table 6.5: Selection of lateral torsional buckling curve for cross sections using equation (6.57)

Cross-section	Limits	Buckling curve
Rolled I-sections	$h/b \le 2$ $h/b > 2$	**b** **c**
Welded I-sections	$h/b \le 2$ $h/b > 2$	**c** **d**
Other cross-sections	-	**d**

(2) For taking into account the moment distribution between the lateral restraints of members the reduction factor χ_{LT} may be modified as follows:

$$\chi_{LT,mod} = \frac{\chi_{LT}}{f} \quad \text{but } \chi_{LT,mod} \le 1 \qquad (6.58)$$

NOTE The values f may be defined in the National Annex. The following minimum values are recommended:

$f = 1 - 0{,}5(1 - k_c)[1 - 2{,}0(\bar{\lambda}_{LT} - 0{,}8)^2]$ but $f \le 1{,}0$

k_c is a correction factor according to Table 6.6

Table 6.6: Correction factors k_c

Moment distribution	k_c
$\psi = 1$	1,0
$-1 \le \psi \le 1$	$\dfrac{1}{1{,}33 - 0{,}33\psi}$
	0,94
	0,90
	0,91
	0,86
	0,77
	0,82

6. EXTENSION TO OTHER CASES.

6.1 Load pattern.

Uniform moment applied to an unrestrained beam is the most severe for consideration of lateral torsional buckling, figure 19.

An elastic analysis of alternative load cases results in higher values of elastic critical moments.

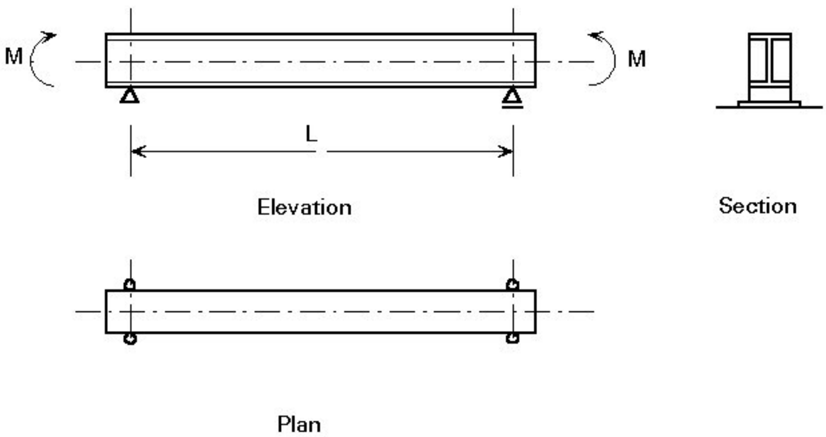


Figure 19 - Definition of basic lateral-torsional buckling problem.

For example, the elastic critical moment for uniform moment is (rearranging eqn (18)):

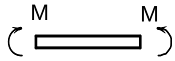
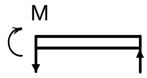
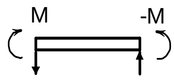
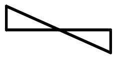
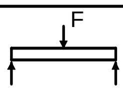
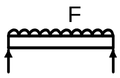
$$M_{cr} = \frac{\pi}{L}\sqrt{EI_z GI_r}\sqrt{1 + \frac{\pi^2 EI_w}{L^2 GI_t}} \qquad (25)$$

but for a beam with a central point load (full derivation in Appendix 1) the maximum moment at the centre on the point of buckling is

$$M_{cr} = \frac{4.24}{L}\sqrt{EI_z GI_r}\sqrt{1 + \frac{\pi^2 EI_w}{L^2 GI_t}} \qquad (26)$$

which is $4.24/\pi$ higher than the base case. EC3 uses this ratio expressed as a factor, C_1, to allow for the loading arrangement (shape of the bending moment diagram) for a variety of loading cases, Figure 20.

C_1 appears as a simple multiplier in expressions for Mcr (EC3 eq. F.2) or as $1/\sqrt{C_1}$ in eqs. for λ_{LT}.

$M_{cr} = C_1 \frac{\pi}{L}\sqrt{EI\,GJ}\sqrt{1+\frac{\pi^2 EI_w}{L^2 GJ}}$			
Beam and loads	Bending moment	M_{max}	C_1
		M	1,00
		M	1,879
		M	2,752
		$\frac{FL}{4}$	1,365
		$\frac{FL}{8}$	1,132

Note: The above values correspond to an effective length factor k of 1,0

Figure 20 - Equivalent uniform moment factors C_1.

If there is a moment gradient between points of lateral restraint, λ_{LT} is calculated as follows:

$$\lambda_{LT} = \frac{L/i_z}{[C_1]^2\left[1+\frac{1}{20}\left(\frac{L/i_z}{d/t_f}\right)^2\right]^{1/4}} \tag{27}$$

where $C_1 = 1{,}75 - 1{,}05\,\psi + 0{,}3\,\psi^2 \leq 2{,}35$ and ψ is the end moment ratio defined in Figure 21.

Figure 21 – Moment gradient loading over beam span L.

Taking as an example the end span of a continuous beam for which $\psi = 0$ gives C_1=1,75 and thus λ_{LT} is reduced to 0,76 (= 1/√ 1,75) of the uniform moment value, leading to an increase in χ_{LT} and thus in M_{bRd}.

BB.3.3 Modification factors for moment gradients in members laterally restrained along the tension flange

BB.3.3.1 Linear moment gradients

(1)B The modification factor C_m may be determined from

$$C_m = \frac{1}{B_0 + B_1\beta_t + B_2\beta_t^{\,2}} \tag{BB.13}$$

in which

$$B_0 = \frac{1+10\eta}{1+20\eta}$$

$$B_1 = \frac{5\sqrt{\eta}}{\pi + 10\sqrt{\eta}}$$

$$B_2 = \frac{0{,}5}{1+\pi\sqrt{\eta}} - \frac{0{,}5}{1+20\eta}$$

$$\eta = \frac{N_{crE}}{N_{crT}}$$

$$N_{crE} = \frac{\pi^2 EI_z}{L_t^{\ 2}}$$

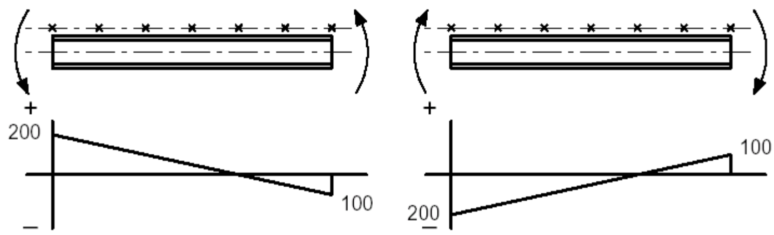
L_t is the distance between the torsional restraints

$N_{crT} = \frac{1}{i_s^{\ 2}}\left(\frac{\pi^2 EI_z a^2}{L_t^{\ 2}} + \frac{\pi^2 EI_w}{L_t^{\ 2}} + GI_t\right)$ is the elastic critical buckling force for an I-section between restraints to both flanges at spacing L_t with intermediate lateral restraints to the tension flange.

$$i_s^2 = i_y^2 + i_z^2 + a^2$$

where a is the distance between the centroid of the member and the centroid of the restraining members, such as purlins restraining rafters

β_t is the ratio of the algebraically smaller end moment to the larger end moment. Moments that produce compression in the non-restrained flange should be taken as positive. If the ratio is less than –1,0 the value of β_t should be taken as –1,0, see Figure BB.4.



$$\beta_t = \frac{-100}{+200} = -0{,}5$$

$$\beta_t = \frac{-200}{+100} = -2$$

but $\beta_t \geq -1{,}0$ thus $\beta_t = -1{,}0$

Figure BB.4: Value of β_t

BB.3.3.2 Non linear moment gradients

(1)B The modification factor C_n may be determined from

$$C_n = \frac{12}{\left[R_1 + 3R_2 + 4R_3 + 3R_4 + R_5 + 2(R_S - R_E)\right]} \qquad \text{(BB.14)}$$

in which R_1 to R_5 are the values of R according to (2)B at the ends, quarter points and mid-length, see Figure BB.5, and only positive values of R should be included.

In addition, only positive values of $(R_S - R_E)$ should be included, where

- R_E is the greater of R_1 or R_5
- R_s is the maximum value of R anywhere in the length L_y

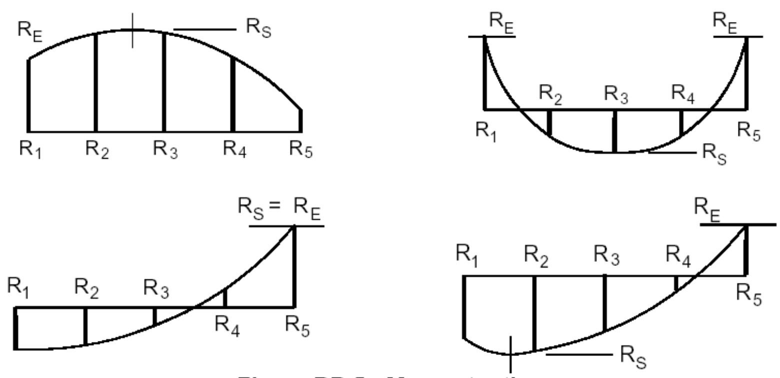


Figure BB.5: Moment ratios

(2)B The value of R should be obtained from:

$$R = \frac{M_{y,Ed} + a\,N_{Ed}}{f_y W_{pl,y}} \qquad \text{(BB.15)}$$

where a is the distance between the centroid of the member and the centroid of the restraining members, such as purlins restraining rafters

6.2 Level of application of load.

Lateral stability of a beam is dependent not only on the arrangement of loads within the span but also on the level of application of the load relative to the centroid.

A similar method to the one presented for different load types can be used to calculate the reduced M_{cr} appropriate for destabilising loads.

These are loads that act above the level of the beam's shear centre and are free to move sideways with the beam as it buckles, as shown in Figure 22.

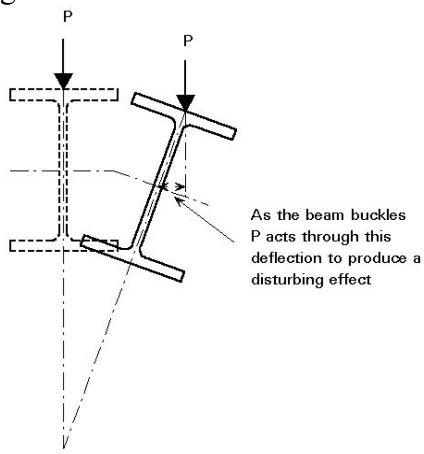


Figure 22 – Destabilising loading.

For cross-sections of the type illustrated in Figure 17c, for which the shear centre and centroid do not lie on the same horizontal axis, evaluation of M_{cr} becomes more complex.

Figure 23 illustrates the effect of placing the load above and below the centroid for a simple span with a central point load.

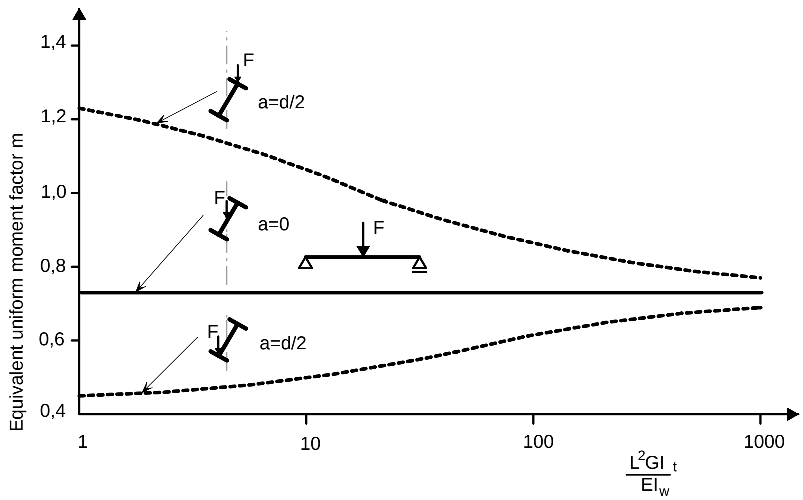


Figure 23 - Effect of level of load application on beam stability

Loads applied to the top flange add to the destabilising effect due to the additional twisting moment arising from the action of the load not passing through the section centroid.

The influence of this behaviour becomes more significant as the depth of the section increases and/or the span reduces ie as L^2GI_t/EI_w becomes smaller.

Again EC3 accounts for this by introduction of a factor C_2 into the general equation for the elastic critical moment (see EC3 eqn F.2) and expressions for λ_{LT} (see EC3 eqns F.27 - F.31).

6.3 End Support Conditions.

All of the foregoing has assumed end conditions which prevent lateral movement and twist but permit rotation on plan.

End conditions which prevent rotation on plan enhance the elastic buckling resistance (in much the same way that column capacities are enhanced by rotational end restraints).

Lateral support arrangements which inhibit the growth of the buckling deformations will improve a beam's lateral stability.

A convenient way of including the effect of different support conditions is to redefine the unrestrained length as an effective length, or more precisely with two effective length factors, k and k_w.

The two factors reflect two possible types of end fixity, lateral bending restraint and warping restraint.

However it should be noted that it is recommended that k_w be taken as 1,0 unless special provision for warping fixing is made.

EC3 recommends k values of 0,5 for fully fixed ends, 0,7 for free & fixed ends and 1,0 for free ends.

The choice of k is at the designer's discretion.

One case of particular practical interest is the cantilever, Figure 24.

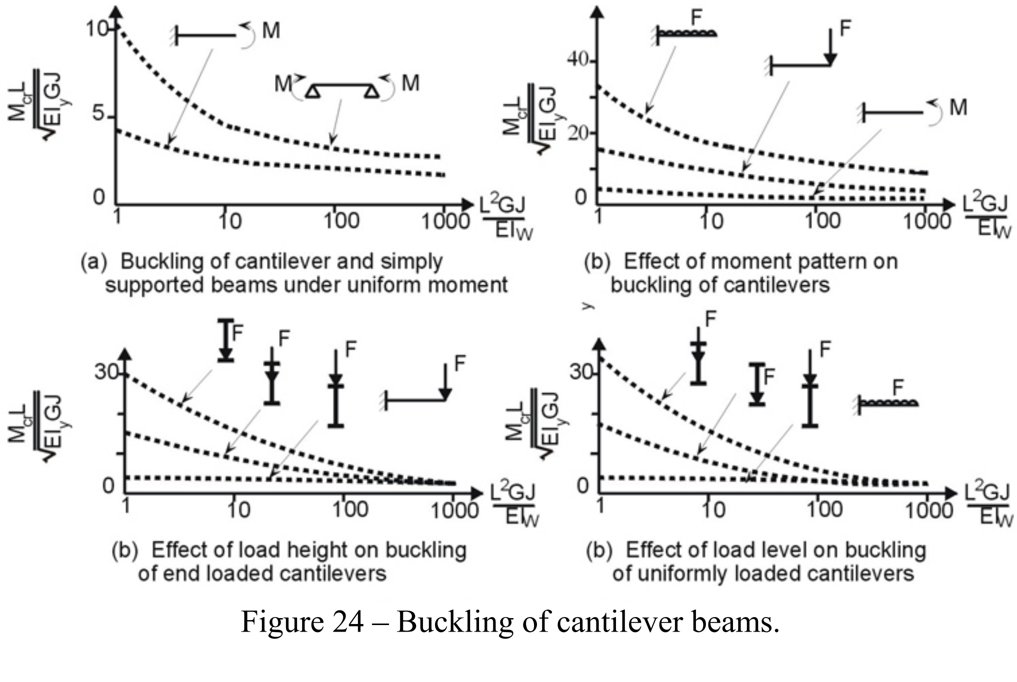


Figure 24 – Buckling of cantilever beams.

These show that:

1. Cantilevers under end moment are less stable than similar, simply supported, beams.
2. Concentrating the moment adjacent to the support, as happens when the applied loading changes from pure moment to an end load or to a distributed load, improves lateral stability.
3. The load height effect is even more significant for cantilevers than for simply supported beams.

6.4 Beams with intermediate lateral support.

Where beams have lateral restraints at intervals along the span the segments of the beam between restraints may be treated in isolation, the design of the beam being based on the most critical segment.

Lengths of beams between restraints should use an effective length factor k of 1,0 <u>not</u> 0,7, as in the buckled shape the adjacent unrestrained length will buckle in sympathy.

6.3.2.4 Simplified assessment methods for beams with restraints in buildings

(1)B Members with discrete lateral restraint to the compression flange are not susceptible to lateral-torsional buckling if the length L_c between restraints or the resulting equivalent compression flange slenderness $\bar{\lambda}_f$ satisfies:

$$\bar{\lambda}_f = \frac{k_c L_c}{i_{f,z}\lambda_1} \leq \bar{\lambda}_{c0}\frac{M_{c,Rd}}{M_{y,Ed}} \tag{6.59}$$

where $M_{y,Ed}$ is the maximum design value of the bending moment within the restraint spacing

$$M_{c,Rd} = W_y \frac{f_y}{\gamma_{M1}}$$

W_y is the appropriate section modulus corresponding to the compression flange

k_c is a slenderness correction factor for moment distribution between restraints, see Table 6.6

$i_{f,z}$ is the radius of gyration of the compression flange including 1/3 of the compressed part of the web area, about the minor axis of the section

$\bar{\lambda}_{c0}$ is the slenderness parameter of the above compression element

$$\lambda_1 = \pi\sqrt{\frac{E}{f_y}} = 93{,}9\varepsilon$$

$$\varepsilon = \sqrt{\frac{235}{f_y}} \qquad (f_y \text{ in N/mm}^2)$$

NOTE 1B For Class 4 cross-sections $i_{f,z}$ may be taken as

$$i_{f,z} = \sqrt{\frac{I_{eff,f}}{A_{eff,f} + \frac{1}{3}A_{eff,w,c}}}$$

where $I_{eff,f}$ is the effective second moment of area of the compression flange about the minor axis of the section

$A_{eff,f}$ is the effective area of the compression flange

$A_{eff,w,c}$ is the effective areas of the compressed part of the web

NOTE 2B The slenderness limit $\bar{\lambda}_{c0}$ may be given in the National Annex. A limit value $\bar{\lambda}_{c0} = \bar{\lambda}_{LT,0} + 0{,}1$ is recommended, see 6.3.2.3.

(2)B If the slenderness of the compression flange $\bar{\lambda}_f$ exceeds the limit given in (1)B, the design buckling resistance moment may be taken as:

$$M_{b,Rd} = k_{f\ell}\chi M_{c,Rd} \quad \text{but} \quad M_{b,Rd} \leq M_{c,Rd} \tag{6.60}$$

where χ is the reduction factor of the equivalent compression flange determined with $\bar{\lambda}_f$

$k_{f\ell}$ is the modification factor accounting for the conservatism of the equivalent compression flange method

NOTE B The modification factor may be given in the National Annex. A value $k_{f\ell} = 1{,}10$ is recommended.

(3)B The buckling curves to be used in (2)B should be taken as follows:

curve d for welded sections provided that: $\frac{h}{t_f} \leq 44\varepsilon$

curve c for all other sections

where h is the overall depth of the cross-section

t is the thickness of the compression flange

NOTE B For lateral torsional buckling of components of building structures with restraints see also Annex BB.3.

6.5 Continuous beams.

Continuity may be present in two different forms:

1. In a beam that has a single span vertically but is subdivided, by intermediate lateral supports, so that it exhibits horizontal continuity between adjacent segments, see Figure 25a.
2. In the vertical plane as illustrated in Figure 25b.

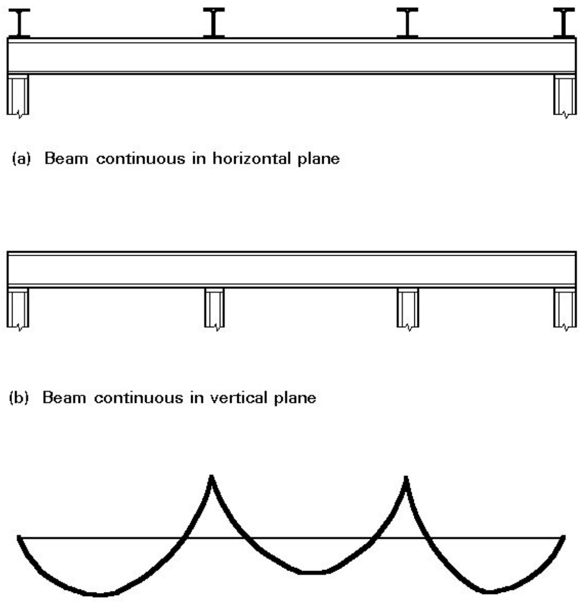


Figure 25 – Two different forms of continuity.

For the first case a safe design will result if the most critical segment, treated in isolation, is used as the basis for designing the whole beam.

For the second case account should be taken of the actual moment diagram within each span, produced by the continuity, by using the C_1 factor.

If the top flange can be considered as laterally restrained because of attachment to a concrete slab, particular attention should be paid to the regions in which the lower flange is in compression, e.g. the support regions or regions where uplift loads can occur.

Beams continuous over a number of spans may be treated as individual spans taking into account the shape of the bending moment diagram within each span as a result of continuity using the C_1 factor.

BB.2 Continuous restraints

BB.2.1 Continuous lateral restraints

(1)B If trapezoidal sheeting according to EN 1993-1-3 is connected to a beam and the condition expressed by equation (BB.2) is met, the beam at the connection may be regarded as being laterally restrained in the plane of the sheeting.

$$S \geq \left(EI_w \frac{\pi^2}{L^2} + GI_t + EI_z \frac{\pi^2}{L^2} 0{,}25h^2 \right) \frac{70}{h^2} \qquad \text{(BB.2)}$$

where S is the shear stiffness (per unit of beam length) provided by the sheeting to the beam regarding its deformation in the plane of the sheeting to be connected to the beam at each rib.

I_w is the warping constant

I_t is the torsion constant

I_z is the second moment of area of the cross section about the minor axis of the cross section

L is the beam length

h is the depth of the beam

If the sheeting is connected to a beam at every second rib only, S should be substituted by $0{,}20S$.

NOTE Eqation (BB.2) may also be used to determine the lateral stability of beam flanges used in combination with other types of cladding than trapezoidal sheeting, provided that the connections are of suitable design.

BB.2.2 Continuous torsional restraints

(1)B A beam may be considered as sufficiently restraint from torsional deformations if

$$C_{\vartheta,k} > \frac{M^2_{pl,k}}{EI_z} K_{\vartheta} K_{\upsilon} \qquad \text{(BB.3)}$$

where $C_{\vartheta,k}$ = rotational stiffness (per unit of beam length) provided to the beam by the stabilising continuum (e.g. roof structure) and the connections

K_{υ} = 0,35 for elastic analysis

K_{υ} = 1,00 for plastic analysis

K_{ϑ} = factor for considering the moment distribution see Table BB.1 and the type of restraint

$M_{pl,k}$= characteristic value of the plastic moment of the beam

(2)B The rotational stiffness provided by the stabilising continuum to the beam may be calculated from

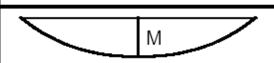
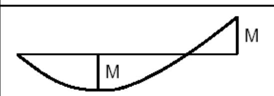
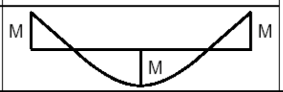
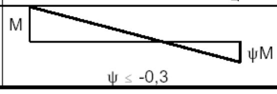
$$\frac{1}{C_{\vartheta,k}} = \frac{1}{C_{\vartheta R,k}} + \frac{1}{C_{\vartheta C,k}} + \frac{1}{C_{\vartheta D,k}} \qquad \text{(BB.4)}$$

where $C_{\vartheta R,k}$ = rotational stiffness (per unit of the beam length) provided by the stabilising continuum to the beam assuming a stiff connection to the member

$C_{\vartheta C,k}$ = rotational stiffness (per unit of the beam length) of the connection between the beam and the stabilising continuum

$C_{\vartheta D,k}$ = rotational stiffness (per unit of the beam length) deduced from an analysis of the distorsional deformations of the beam cross sections, where the flange in compression is the free one; where the compression flange is the connected one or where distorsional deformations of the cross sections may be neglected (e.g. for usual rolled profiles) $C_{\vartheta D,k} = \infty$

Table BB.1: Factor K_{ϑ} for considering the moment distribution and the type of restraint

Case	Moment distribution	without translational restraint	with translational restraint
1		4,0	0
2a		3,5	0,12
2b			0,23
3		2,8	0
4		1,6	1,0
5		1,0	0,7

6.6 Beams Other than Doubly-Symmetrical I-sections.

Equation (17) is valid for members symmetrical about their horizontal axis, i.e. a channel with the web vertical, providing the moments act through the shear centre (which will not coincide with the centroid).

However, sections symmetrical only about the vertical axis, e.g. an unequal flanged I, require some modification so as to allow for the so-called Wagner effect.

This arises as a direct result of the vertical separation of the shear centre and the centroid and leads to either an increase or a decrease in the section's torsional rigidity.

Thus lateral stability will be improved (compared with equal flange sections) when the larger flange is in compression and vice versa.

Sections with no axis of symmetry will not actually buckle but will deform by bending about both principal axes and by twisting from the onset of loading and should be treated in the same way as symmetrical sections under biaxial bending.

6.7 Restrained Beams.

The elastic critical moment for a doubly symmetrical I-beam provided with continuous elastic torsional restraint, of stiffness equal to K_ϕ , is:

$$M_{cr} = \left(\pi/L\right)\sqrt{EI_z GI_t}\sqrt{1+\left(\frac{\pi^2 EI_w}{L^2 GI_t}\right)+\frac{K_\phi L^2}{\pi^2 GI_t}} \tag{17}$$

Rearranging this shows that the beam behaves as if its torsional rigidity GI_t were increased to $(GI_t + K_\phi L^2/\pi^2)$, thereby permitting a ready assessment of the effectiveness of the restraint.

An important practical example of such a restraint would be that provided by the bending stiffness of profiled steel sheeting (used typically in roof construction) spanning at right angles to the beam.

7. CONCLUDING SUMMARY

- Beams bent about the major axis may fail in a more flexible plane - lateral-torsional buckling
- The elastic critical moment which causes lateral-torsional buckling of a slender beam may be determined from an analysis which has close similarities to that used to study column buckling.
- Moment at which buckling occurs is the elastic critical moment
- Bracing of sufficient stiffness and strength, that restrains either torsional or lateral deformations, may be used to increase buckling resistance.
- Design approach must account for a large number of factors -
  section shape, the degree of lateral restraint, type of loading,
  residual stress pattern and initial imperfections
- Stocky beams are unaffected by lateral torsional buckling
- Slender beams have capacities close to the theoretical elastic critical moment
- Practical beams are significantly adversely affected by inelasticity and geometrical imperfections - elastic theory is an upper band solution.
- A design expression linking the plastic capacity of stocky beams with the elastic behaviour of slender beams is provided by a reduction factor for lateral torsional buckling, χ_{LT}
- Examination of the expression for the elastic critical moment for the basic problem enables the influence of cross-sectional shape, as it affects the beam's resistance to lateral bending (EI_z), torsion (I_t) and warping (I_w), to be identified; it also demonstrates the importance of unbraced span length.
- Variation in either lateral support conditions or the form of the applied loading may be accommodated in the design process by means of coefficients k and C, used to modify either the basic slenderness λ_{LT} or the basic elastic critical moment M_{cr}.
- Extensions to the basic theory permit the effects of load pattern, end restraint and level of application of destabilising loads to be quantified.

- Load patterns which produce non-uniform moment may be compared with the uniform moment case using the coefficient C_1; most of these cases will be less severe producing values greater than 1.

ADDITIONAL READING

1. Narayanan, R., Editor, "Beams and Beam Columns: Stability and Strength", Applied Science Publishers 1983.
2. Chen, W. F. and Atsuta, T. "Theory of Beam Columns Volume 2, Space Behaviour and Design", McGraw Hill 1977.
3. Timoshenko, S. P. and Gere, J. M., "Theory of Elastic Stability" 2nd Edition, McGraw Hill 1961.
4. Bleich, F., "Buckling Strength of Metal Structures", McGraw Hill 1952.
5. Galambos, T. V., "Structural Members and Frames", Prentiss Hall 1968.
6. Trahair, N. S. and Bradford, M. A., "The Behaviour and Design of Steel Structures", Chapman and Hall, Second Edition, 1988.

APPENDIX 1: BUCKLING OF A CENTRALLY LOADED BEAM

Consider the beam subjected to a central load acting at the level of the centroidal axis in Figure 26.

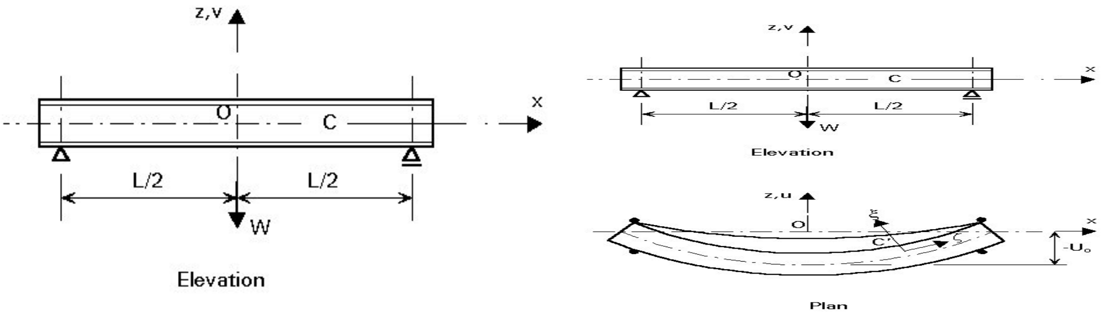


Figure 26 – Buckling of a beam with a central transverse load.

Noting from Figure 10 that the vertical load will produce a moment about the x-axis of $W(u_o - u)/2$ when the beam is in its buckled position, enables Equations (4) to be re-written as:

$$M_\xi = \frac{W}{2}\left(\frac{L}{2}-x\right)$$

$$M_\eta = \frac{W}{2}\left(\frac{L}{2}-x\right)\sin\phi \qquad \text{(A1)}$$

$$M_\zeta = \frac{W}{2}\left(\frac{L}{2}-x\right)\sin\alpha = \frac{W}{2}(u_0-u)\cos\alpha$$

Replacing Equations (5) - (7) by their revised forms and eliminating u from the 2nd and 3rd of these:

$$EI_w\frac{d^4\phi}{dx^4} - GI_t\frac{d^2\phi}{dx^2} - \frac{W^2}{4EI_z}\left(\frac{L}{2}-x\right)^2\phi = 0 \qquad \text{(A2)}$$

which may be solved for W_{cr} to yield approximately:

$$W_{cr} = 5.4\left(\pi/L^2\right)\sqrt{EI_zGI_t}\sqrt{1+\left(\frac{\pi^2EI_w}{L^2GI_t}\right)} \qquad \text{(A3)}$$

The moment at midspan is then:

$$M_{cr} = \frac{4.24}{L}\sqrt{EI_zGI_t}\sqrt{1+\left(\frac{\pi^2EI_w}{L^2GI_t}\right)} \qquad \text{(A4)}$$

The alternative means of obtaining elastic critical loads uses the energy method, in which the work done by the applied load during buckling is equated to the additional strain energy stored as a result of the buckling deformations. Considering an element of the longitudinal axis of the beam of length dx located at C, bending in the ξ ζ plane causes the end B of the beam to rotate in the ξ ζ plane by:

$$\frac{d^2u}{dx^2}\left(\frac{L}{2}-x\right)dx \qquad \text{(A5)}$$

The vertical component of this is:

$$\phi \frac{d^2u}{dx^2}\left(\frac{L}{2} - x\right)dx \tag{A6}$$

Summing these for all elements between x= 0 and x = L/2 gives the lowering of the load W from which the work is:

$$W\int_0^{L/2} \phi \frac{d^2u}{dx^2}\left(\frac{L}{2} - x\right)dx \tag{A7}$$

The strain energy stored as a result of lateral bending, twisting and warping is:

$$\frac{EI_z}{2}\int_0^L \left[\frac{d^2u}{dx^2}\right]^2 dx + \frac{EI_w}{2}\int_0^L \left[\frac{d^2\phi}{dx^2}\right]^2 dx + \frac{GI_T}{2}\int_0^L \left[\frac{d\phi}{dx}\right]^2 dx \tag{A8}$$

Assuming a buckled shape of the form:

$$\phi = a_1 \cos\frac{\pi x}{L} + a_2 \cos\frac{3\pi x}{L} \tag{A9}$$

and equating Equations (A7) and (A8) enables the critical value of W to be obtained. This technique permits examination of cases where the load is applied at a level other than the centroidal axis. Assuming W to act at a vertical distance (a) above the centroid, the additional work will be:

$$Wa(1 - \cos\phi_0) = Wa\,\phi_0^2/2 \tag{A10}$$

in which ϕ_o is the value of ϕ at the load point. This must be added to Equation (A7).