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THIN BEAM STATIC STABILITY ANALYSIS BY AN IMPROVED NUMERICAL METHOD

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Abstract. *The finite volume meshless local Petrov-Galerkin (FVMLPG) method is a new meshless method for the discretization of governing differential equations. The motivation for developing this new method is to unify advantages of meshless methods and finite volume methods (FVM) in one scheme. The purpose of this paper is to develop and use of the FVMLPG method for static stability of thin beam problems. In this method, the weak formulation of a conservation law is discretized by restricting it to a discrete set of test functions. In contrast to the usual Finite Volume approach, the test functions are not taken as characteristic functions of the control volumes in a spatial grid, but are chosen from a Heaviside step function. The present approach eliminates the expensive process of directly differentiating the MLS interpolations in the entire domain. This method was evaluated by applying the formulation to a thin beam problem. The formulation successfully reproduced exact solutions. Numerical example demonstrates the present method yields accurate results for the critical loads.*

1 INTRODUCTION

The finite volume meshless local Petrov-Galerkin (FVMLPG) method is a new meshless method for the discretization of governing differential equations. The motivation for developing this new method is to unify advantages of meshless methods and finite volume methods (FVM) in one scheme. The basic idea in the FVMLPG is to incorporate elements of the FVM into a meshless local Petrov-Galerkin (MLPG) method [1,2].

Meshless methods are very flexible because they are free of using mesh. The need for meshless methods typically arises if problems with time dependent or very complicated geometries are under consideration because the handling of mesh discretization becomes technically complicated or very time consuming. Fluid flows with structural interaction or fast moving boundaries like an inflating air-bag are of that kind for instance.

Advantages of meshless methods are to overcome some of the disadvantages of mesh-based methods such as discontinuous secondary variables across inter-element boundaries and the need for remeshing in large deformation problems [3-7]. Extensive research on meshless methods, in particular, the meshless local Petrov-Galerkin (MLPG) method recently exists in literatures. There is analysis of thin beam problems using a Galerkin implementation of the MLPG method [9]; a generalized moving least squares (GMLS) approximation is used to construct the trial functions, and the test functions are chosen from the same space. References [8] and [9] showed good performance of the MLPG method for potential and elasticity problems and a good performance for beam problems. However, these methods need a large number of calculations to compute the first and second order derivatives of the moving least squares (MLS) trial functions that are required in the weak form and special procedures were needed to integrate the weak form accurately.

The purpose of this paper is to develop and use of the FVMLPG method for static stability of thin beam problems. The method is evaluated by applying the formulation to an example.

The outline of the paper is as follows. First, the FV form of the governing differential equation is derived in a general sense, and a system of algebraic equations is developed from this FV form. Next, the MLPG method is used to descritize these formulations and to obtain the FVMLPG form of the governing differential equation. Finally, the performance of the FVMLPG method is investigated by implying to an example.

2 MESHLESS INTERPOLATION

In general, meshless methods use a local interpolation, or an approximation, to represent the trial function, using the values (or the fictitious values) of the unknown variable at some randomly located nodes in the local vicinity. The moving least-square method is generally considered to be one of the best schemes to interpolate data with a reasonable accuracy. Basically the MLS interpolation does not pass through the nodal data. Consider a domain in question with control points for boundaries (i.e. nodes on boundaries) and some scattered nodes inside, where every node has its undetermined nodal coefficient (fictitious nodal value) and an influence radius (radius for local weight function). Now for the distribution of trial function at any point x and its neighborhood Ω_s located in the problem domain Ω, $u^h(x)$ may be defined by

$$u^h(x) = \mathrm{p}^T(x)\,\mathrm{a}(x) \qquad \forall x \in \Omega_s \tag{1}$$

where $\mathrm{p}^T(x) = [p_1(x), p_2(x), \ldots, p_m(x)]$ is a complete monomial basis of order m, and $\mathbf{a}(x)$ is a vector containing coefficients $a_j(x)$, j=1, 2, . . . , m which are functions of the space co-ordinates x. The commonly used bases in 1-D are the linear basis (m=2), due to their simplicity. In the present 4th order problem, we will also employ the quadratic basis (m=3)

$$\mathrm{p}^T(x) = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \tag{2}$$

and the cubic basis (m=4)

$$\mathrm{p}^T(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \tag{3}$$

The coefficient vector $\mathbf{a}(x)$ is determined by minimizing a weighted discrete L_2 norm, which can be defined as

$$J(x) = \sum_{I=1}^{N} w_I(x)\left[\mathrm{p}(x_I)\,\mathrm{a}(x) - \hat{u}^I\right]^2 \tag{4}$$

where $w_I(x)$, is a weight function associated with the node I, with $w_I(x) > 0$ for all x in the support of $w_I(x)$, x_I denotes the value of x at node I, N is the number of nodes in Ω_s for which the weight functions $w_I(x) > 0$. Here it should be noted that $\hat{u}_I$, I=1, 2, . . . , N, in equation (4), are the fictitious nodal values (undetermined nodal coefficients), and not the exact nodal values of the unknown trial function $u^h(\mathbf{x})$, in general.

Solving for $\mathbf{a}(x)$ by minimizing J in equation (4), and substituting it into equation (1), give a relation which may be written in the form of an interpolation function similar to that used in the FEM, as

$$u^h(x) = \sum_{I=1}^{N} \phi^I(x)\hat{u}^I \qquad u^h(x_I) \equiv u^I \neq \hat{u}^I,\; x \in \Omega_s \tag{5}$$

where

$$\phi^I(x) = \sum_{j=1}^{m} p_j(x)\left[\mathrm{A}^{-1}(x)\mathrm{B}(x)\right]_{jI} \tag{6}$$

with the matrix $\mathbf{A}(x)$ and $\mathbf{B}(x)$ being defined by

$$\mathrm{A}(x) = \sum_{I=1}^{N} w_I(x)\mathrm{p}(x_I)\mathrm{p}^T(x_I) \tag{7}$$

$$\mathrm{B}(x) = \left[w_1(x)\mathrm{p}(x_1), w_2(x)\mathrm{p}(x_2), \ldots, w_N(x)\mathrm{p}(x_N)\right]. \tag{8}$$

The nodal shape function is complete up to the order of the basis. The smoothness of the nodal shape function $\Phi^I(x)$ is determined by that of the basis and of the weight function. The choice of the weight function is more or less arbitrary as long as the weight function is positive and continuous. The following weight function is considered in the present work

$$w_I(\mathrm{x}) = \begin{cases} 1-6\left(\dfrac{d_I}{r_I}\right)^2 + 8\left(\dfrac{d_I}{r_I}\right)^3 - 3\left(\dfrac{d_I}{r_I}\right)^4 & 0 \le d_I \le r_I = \rho_I h_I \\ 0 & d_I > r_I = \rho_I h_I \end{cases} \tag{9}$$

where $d_I = |x - x_I|$ is the distance from node x_I to point x, h_I in the nodal distance, ρ_I is the scaling parameter for the size of the subdomain $\Omega^I_{tr.}$

3 FVMLPG APPROACH

Consider a thin beam as shown in figure 1. The governing equation of an Euler-Bernoulli beam under a compressive axial force N is written as, [10]

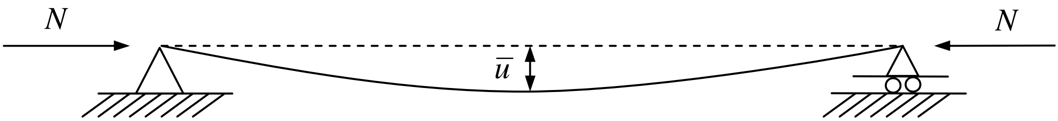

Figure 1: Transverse deformation of a thin beam under axial compression

$$\frac{d^4u}{dx^4}+N\frac{d^2u}{dx^2}=0 \quad \text{in global domain } \Omega \qquad \text{(adimensional form)} \tag{10}$$

Where u is transverse displacement of the beam.

We use the FVMLPG method to solve the homogeneous differential equation (10). The local weak form of Eq (10) can be obtained by multiplying a test function in this equation and integrating over subdomains

$$\int_{\Omega_s}\left(\frac{d^4u}{dx^4}+N\frac{d^2u}{dx^2}\right)vdx=0 \tag{11}$$

To obtain an accurate and efficient meshless method, one should decrease the order of the derivatives of the trial function in the local weak forms. Now a FVMLPG method is presented by redefining the original problem, Eq. (10), in terms of four first-order equations, with the variables Φ_i, (i=1, … , 4) as

$$u=\phi_1\,,\ \frac{d\phi_1}{dx}=\phi_2\,,\ \frac{d\phi_2}{dx}=\phi_3\,,\ \frac{d\phi_3}{dx}=\phi_4\,,\ \frac{d\phi_4}{dx}=1 \tag{12}$$

In matrix notation, Eq.(11) can be written as the following form

$$\begin{Bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{Bmatrix}\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}'+\begin{Bmatrix}0&-1&0&0\\0&0&-1&0\\0&0&0&-1\\0&0&0&0\end{Bmatrix}\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\1\end{Bmatrix}$$

or $$A_{ij}\phi'_j+B_{ij}\phi_j=f_i \tag{13}$$

Each of Φ_j is interpolated through an MLS scheme, as

$$\phi_j(x)=\sum_{I=1}^{N}\phi_j^I(x)\hat{\phi}_j^I\,; \qquad x\in\Omega_s \tag{14}$$

Using a test function which is a Heaviside Step function in each Ω_s [i.e., $x_{k-1} \leq x \leq x_k$, which may be overlapping subdomains as in the finite volume method], the following simple linear equation system is obtained for the nodal values of $\hat{\phi}_j^I$

$$\int_{x_{k-1}}^{x_k} \left[A_{ij}\phi'_j + B_{ij}\phi_j \right] dx = \int_{x_{k-1}}^{x_k} \left[B_{ij}\phi_j \right] dx + \left[A_{ij}\phi_j \right]_{x_{k-1}}^{x_k} = 0 \tag{15}$$

Equation (11) involves only the MLS interpolations for each Φ_i, (j=1, …,4), but not their derivatives. Thus, the Finite Volume Meshless Local Petrov-Galerkin (FVMLPG) method presented in this article is entirely analogous to the finite volume method [11-13]. This equation may also be written in discrete form

$$\sum_{I=1}^{N} \left(\int_{x_{k-1}}^{x_k} B_{ij}\phi_j^I(x)\, dx + \left[A_{ij}\phi_j^I(x) \right]_{x_{k-1}}^{x_k} \right) \hat{\phi}_j^I = 0 \tag{16}$$

The linear system of equations are obtained for the first term, as

$$\frac{d^4 u}{dx^4} = K_{ij} a_j \tag{17}$$

where K_{ij} is defined by

$$K_{ij} = \sum_{I=1}^{N} \left(\int_{x_{k-1}}^{x_k} B_{ij}\phi_j^I(x)\, dx + \left[A_{ij}\phi_j^I(x) \right]_{x_{k-1}}^{x_k} \right) \tag{18}$$

with the same approach, we may obtain the similar equations for the second term, as

$$N \frac{d^2 u}{dx^2} = N C_{ij} a_j \tag{19}$$

By adding the equations (17) and (19), we may obtain the system equations for the linear buckling problems, as

$$\left(K_{ij} + N C_{ij} \right) a_j = 0 \tag{20}$$

Equation (10) is a generalized eigenvalue problem. Its nontrivial solutions, i.e. the eigenvalues and the corresponding eigenvectors $\left(N^i, a_j \right)$, are the critical loads and the corresponding buckling modes, respectively. The procedure of calculation for this method is implemented in a computer code developed in MATLAB. The developed code was validated on a certain number of test cases by comparison with analytical solutions. The buckling mode shapes are shown in figures 2, 3, 4 and 5. They agree with the analytical solution very well. The eigenvalues of the problem are represented in the following table.

Table 1 Four Eigenvalues of simply supported beam

Eigenvalue	1	2	3	4
FVMLPG	39.4816	157.9342	355.3735	631.8252
Exact	39.4784	157.9137	355.3058	631.6547
Error %	0.008	0.013	0.019	0.027

As indicated in Table 1, the method can obtain good results in eigenvalues for the bucking of the simply supported thin beam.

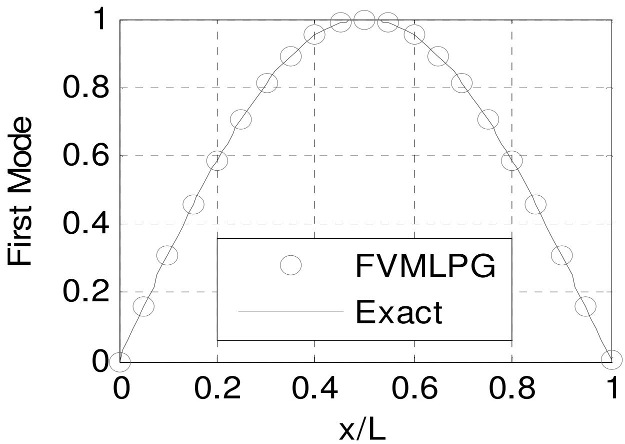

Figure 2: The first buckling mode shapes of the simply supported beam

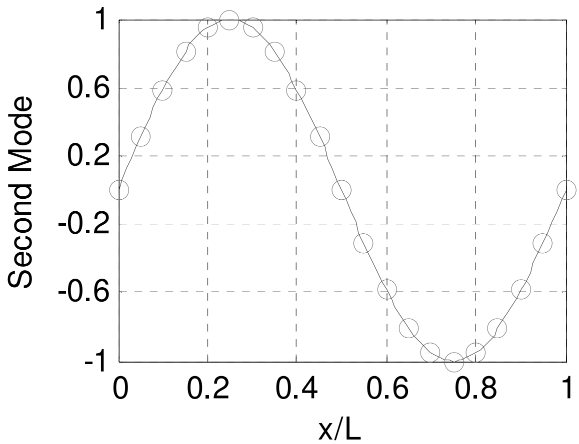

Figure 3: The second buckling mode shapes of the simply supported beam

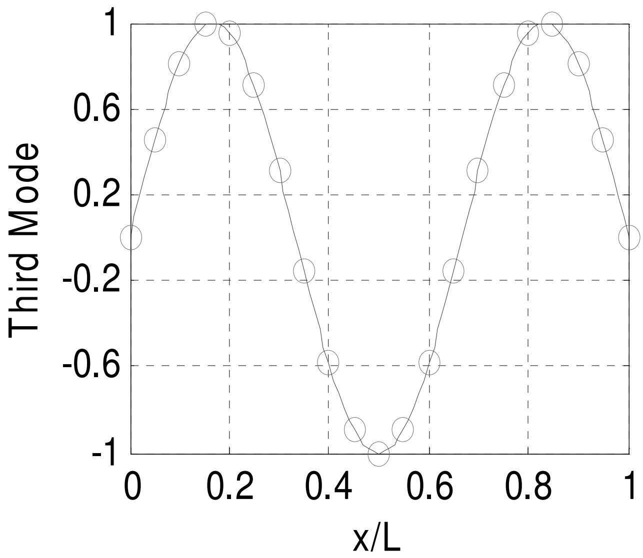

Figure 4: The third buckling mode shapes of the simply supported beam

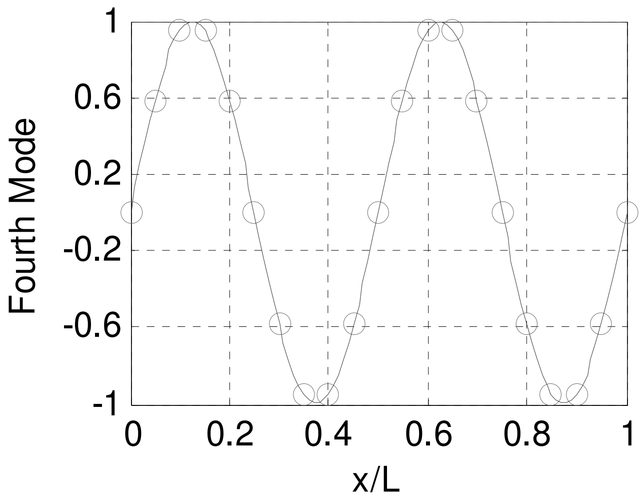

Figure 5: The four buckling mode shapes of the simply supported beam

4 CONCLUSION

This paper presented the FVMLPG method applied to thin beam stability analysis. The FVMLPG method unifies the major advantages of meshless methods and finite volume methods in one single scheme. In the local weak form (LWF) of the governing differential equation, a moving least squares (MLS) interpolation was used to form the approximations to the solution known as trial functions. Test functions, also needed for the LWF were chosen from a different space than the trial functions, making the method a Petrov-Galerkin method. This choice of test functions led to unsymmetric stiffness matrices. The essential boundary conditions were enforced by a collocation method, and numerical integration was used to evaluate the integrals in the system matrices. With the FVMLPG, it is not necessary to differentiate the shape function. In addition, the continuity requirement on the trial function reduces by three-order then it is possible to use a smaller support size in the meshless approximations with a lower-order polynomial basis. The FVMLPG method was applied to and passed several patch test problems. Very good results for both the variables were obtained. A smooth distribution of the secondary variable was obtained without the use of elaborate post processing techniques.

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