SDSS'Rio 2010 STABILITY AND DUCTILITY OF STEEL STRUCTURES
E. Batista, P. Vellasco, L. de Lima (Eds.)
Rio de Janeiro, Brazil, September 8 - 10, 2010

BUCKLING OF A SHALLOW RECTANGULAR BIMETALLIC SHELL SUBJECTED TO OUTER LOADS AND TEMPERATURE

M. Jakomin*, F. Kosel**

* Faculty of Maritime studies and Transport, University of Ljubljana, Pot pomorščakov 4, 6320 Portorož, Slovenia

** Faculty of Mechanical engineering, University of Ljubljana, Aškerčeva 6, 1000 Ljubljana, Slovenia
e-mails: marko.jakomin@fpp.uni-lj.si, franc.kosel@fs.uni-lj.si

Keywords: bimetallic shell, translation shell, temperature load, snap-through of the system, large displacement theory, moderate rotation

Abstract. *In the article, we have formulated a geometric non-linear mathematical-physical model of the snap-through of the system of a thin-walled shallow bimetallic translation shell in a homogenous temperature field according to the theory of large displacements, moderate rotations, and small strains of the shell element. The model enables the calculation of the geometric conditions, of shallow translation shells, due to the influences of temperature and mechanical loads. The results are based on the numeric solution of a non-linear system of partial differential equations with boundary conditions according to the finite difference method.*

1. INTRODUCTION

In practice bimetallic line and plane elements with different coefficients of linear temperature expansion are used in a range of machines and devices. They are mostly used as safety constructional elements against temperature overheating of these machines and devices. They are also used for different purposes such as thermo-elements, blinkers, and for temperature measurement. The function of a bimetallic construction element is based on the physical fact that bodies expand with the increase of temperature. Ideally, homogenous bodies expand and contract isotropically. In the case of bimetallic bodies manufactured from two materials with different temperature expansion coefficients the deformations due to temperature changes are not isotropic. This study discusses the stress and deformation conditions for a thin double curved bimetallic translation shell, which due to the possibilities of constructing different curvatures into the longitudinal and transversal directions enables different relations between the upper and lower temperature snap-through. We also took into consideration nonlinear terms in the deformation tensor, while we placed equilibrium equations on the deformed element of the bimetallic shell.

2. THERMOELASTIC EQUATIONS OF THE PROBLEM

On the element of a deformed shell that is created by cutting the shell in the direction of the curvilinear coordinates $d\overline{s}_1$ and $d\overline{s}_2$ we observe equilibrium of all forces and moments. In this way five equations can be derived for the equilibrium of forces and moments on a deformed body [1,2]:

$$\frac{\partial}{\partial x_1}\left(\overline{A}_2\,\vec{F}_1\right)+\frac{\partial}{\partial x_2}\left(\overline{A}_1\,\vec{F}_2\right)=0 \qquad \frac{\partial}{\partial x_1}\left(\overline{A}_2\,\vec{\Omega}_1\right)+\frac{\partial}{\partial x_2}\left(\overline{A}_1\,\vec{\Omega}_2\right)+\overline{A}_1\,\overline{A}_2\left(\vec{\overline{e}}_1\times\vec{F}_1+\vec{\overline{e}}_2\times\vec{F}_2\right)=0$$

Then the system of equilibrium equations is supplemented with three kinematic equations, three constitutive equations and in addition also with eight equations for the forces and moments per unit of length [3,4,5,6]. Thus, we have obtained the system of thermo-elastic equations, which consists of the 19[th] equations and of the same number of unknown variables [2]. However, the number of equations and unknowns in this system can be reduced by proper substitution. Finally a geometric non-linear mathematical-physical model of the snap-through of the system of a thin-walled shallow bimetallic translation shell in a homogenous temperature field according to the theory of large displacements, moderate rotations, and small strains of the shell element is achieved as a system of three non-linear partial differential equations (1) with appurtenant boundary conditions (2) where displacements u, v and w act as unknown functions [1,2]:

$$\boxed{\begin{aligned}
N_1\Big|_{x_1=a} &= \left(D\left(u' + w\,y_1'' + \frac{1}{2}\left(w'\right)^2\right) + \bar{D}\left(\dot{v} + w\,\ddot{y}_2 + \frac{1}{2}\left(\dot{w}\right)^2\right) + G\,T\right)\Bigg|_{x_1=a} = 0\\
T_{12}^R\Big|_{x_1=a} &= \left(H\left(v' + \dot{u} + w'\,\dot{w}\right) + 2L\,\dot{w}'\left(\ddot{y}_2 - \ddot{w}\right)\right)\Big|_{x_1=a} = 0\\
T_{13}^R\Big|_{x_1=a} &= J\,w''' + \bar{J}\,\ddot{w}' + 4L\left(\ddot{w}'\right)\Big|_{x_1=a} = -\left(q_3\,b + c_2\right)\frac{a}{b}\\
M_1\Big|_{x_1=a} &= \left(J\,w'' + \bar{J}\,\ddot{w} + K\,T\right)\Big|_{x_1=a} = 0\\
N_2\Big|_{x_2=b} &= \left(\bar{D}\left(u' + w\,y_1'' + \frac{1}{2}\left(w'\right)^2\right) + D\left(\dot{v} + w\,\ddot{y}_2 + \frac{1}{2}\left(\dot{w}\right)^2\right) + G\,T\right)\Bigg|_{x_2=b} = 0\\
T_{21}^R\Big|_{x_2=b} &= \left(H\left(v' + \dot{u} + w'\,\dot{w}\right) + 2L\,\dot{w}'\left(y_1'' - w''\right)\right)\Big|_{x_2=b} = 0\\
T_{23}^R\Big|_{x_1=b} &= J\,\dddot{w} + \bar{J}\,\dot{w}'' + 4L\left(\dot{w}''\right)\Big|_{x_1=b} = -\left(q_3\,a + c_1\right)\frac{b}{a}\\
M_2\Big|_{x_2=b} &= \left(J\,\ddot{w} + \bar{J}\,w'' + K\,T\right)\Big|_{x_2=b} = 0
\end{aligned}} \tag{1}$$

$$\boxed{\begin{aligned}
N_1\Big|_{x_1=-a} &= \left(D\left(u' + w\,y_1'' + \frac{1}{2}\left(w'\right)^2\right) + \bar{D}\left(\dot{v} + w\,\ddot{y}_2 + \frac{1}{2}\left(\dot{w}\right)^2\right) + G\,T\right)\Bigg|_{x_1=-a} = 0\\
T_{12}^R\Big|_{x_1=-a} &= \left(H\left(v' + \dot{u} + w'\,\dot{w}\right) + 2L\,\dot{w}'\left(\ddot{y}_2 - \ddot{w}\right)\right)\Big|_{x_1=-a} = 0\\
T_{13}^R\Big|_{x_1=-a} &= J\,w''' + \bar{J}\,\ddot{w}' + 4L\left(\ddot{w}'\right)\Big|_{x_1=-a} = \left(q_3\,b + c_2\right)\frac{a}{b}\\
M_1\Big|_{x_1=-a} &= \left(J\,w'' + \bar{J}\,\ddot{w} + K\,T\right)\Big|_{x_1=-a} = 0\\
N_2\Big|_{x_2=-b} &= \left(\bar{D}\left(u' + w\,y_1'' + \frac{1}{2}\left(w'\right)^2\right) + D\left(\dot{v} + w\,\ddot{y}_2 + \frac{1}{2}\left(\dot{w}\right)^2\right) + G\,T\right)\Bigg|_{x_2=-b} = 0\\
T_{21}^R\Big|_{x_2=-b} &= \left(H\left(v' + \dot{u} + w'\,\dot{w}\right) + 2L\,\dot{w}'\left(y_1'' - w''\right)\right)\Big|_{x_2=-b} = 0\\
T_{23}^R\Big|_{x_1=-b} &= J\,\dddot{w} + \bar{J}\,\dot{w}'' + 4L\left(\dot{w}''\right)\Big|_{x_1=-b} = \left(q_3\,a + c_1\right)\frac{b}{a}\\
M_2\Big|_{x_2=-b} &= \left(J\,\ddot{w} + \bar{J}\,w'' + K\,T\right)\Big|_{x_2=-b} = 0
\end{aligned}} \tag{2}$$

3. NUMERIC SOLUTION TO THE SISTEM

We solved the system of non-linear differential equations (1) with the boundary conditions (2) by using a finite difference method [6,7] in the program package Mathematica 7.0.0. Below are the results for a shell loaded with a temperature T and with a force per unit area q_3. This force is compensated on the shell's edges by the constant transversal forces T_{13}^R and T_{23}^R:

$$T_{13}^R\Big|_{x_1=a} = T_{23}^R\Big|_{x_2=b} = \frac{-q_3\,a}{2} = \frac{-F}{8\,a} \tag{3}$$

The shell has the following material and geometric characteristics:

$$\begin{array}{|llll|}
\text{construction curves:} & & y_1 = \dfrac{0,5}{10^2 mm}x_1^2 & y_2 = \dfrac{0,5}{10^2 mm}x_2^2 \\
\text{ground plan radii:} & a = b = 10\,mm & & \\
\text{thickness:} & & \delta = 2\delta_1 = 2\delta_2 = 0,3\,mm & \\
\text{Young's modulus:} & E_1 = E_2 = E = \dfrac{170000N}{mm^2} & & \\
\text{Poisson's ratio:} & & \mu_1 = \mu_2 = \mu = 1/3 & \\
\text{temperature expansion:} & & \alpha_1 = 3,41\cdot 10^{-5}K^{-1} & \alpha_2 = 1,41\cdot 10^{-5}K^{-1}
\end{array} \tag{4}$$

We slowly heat up a shell that is loaded with an equally distributed force $F = -20\,N$ along the upper surface of the shell. So the force per unit area equals $q_3 = -0,005\,N/mm^2$. Let us observe the change in the relations of heights ξ in dependence of temperature T.

$$\xi = \frac{h(x_1)}{h_0(x_1)}\Bigg|_{x_1=a} = \frac{Y_1(a)}{y_1(a)} = \frac{y_1(a) - w(a)}{y_1(a)} = 1 - \frac{w(a)}{y_1(a)} \tag{5}$$

The stability curves in dependence from mechanical loads q_3 and temperature T for a shell with the geometric characteristics in (4) are shown in Figure 1. With the increase of force per unit area q_3, the temperature of both snap-through T_{p1} and T_{p2} decreases.

Table 1: The snap-through temperatures in dependence from the load q_3 in case of an equal support on the shell edges

$q_3 = \frac{F}{4a^2}\left[N/mm^2\right]$	$F = 0$ $q_3 = 0$	$F = -20$ $q_3 = -0,05$	$F = -40$ $q_3 = -0,1$	$F = -70$ $q_3 = -0,175$	$F = -105$ $q_3 = -0,2625$
$T_{p1}\left[C^\circ\right]$	$T_{p1} = 102,4$ $\xi_1 = 0,20$	$T_{p1} = 82,2$ $\xi_1 = 0,28$	$T_{p1} = 61,3$ $\xi_1 = 0,32$	$T_{p1} = 31,4$ $\xi_1 = 0,35$	$T_{p1} = -3,7$ $\xi_1 = 0,34$
$T_{p2}\left[C^\circ\right]$	$T_{p2} = 97,6$ $\xi_2 = 0,20$	$T_{p2} = 77,6$ $\xi_2 = -0,28$	$T_{p2} = 55,3$ $\xi_2 = -0,27$	$T_{p2} = 24,4$ $\xi_2 = -0,28$	$T_{p1} = -11,8$ $\xi_2 = -0,27$

When the force per unit area q_3, is strong enough the shell will snap-through without additional heating. With interpolation of the snap-through temperature T_{p1} in dependence from external loads q_3 in

Table 1, we have calculated, in the treated case, that the shell without additional heating snaps-through if loaded with an equally distributed force $F = -101.4\,N$ along the upper surface of the shell, which amounts to a mechanical load of $q_3 = -0.2535\,N/mm^2$. The flat state of the bimetallic shell occurs at the temperature T_f. If the shell with two equal parabolic construction curves is loaded only with temperature T, then T_f can be readily calculated from the boundary condition (2) for the bending moment M_1 or M_2 from which the temperature T_f of the flat shell follows:

$$T_f = \frac{-2k\left(J+\bar{J}\right)}{K} = \frac{4k\delta}{3\left(\alpha_1-\alpha_2\right)} \tag{6}$$

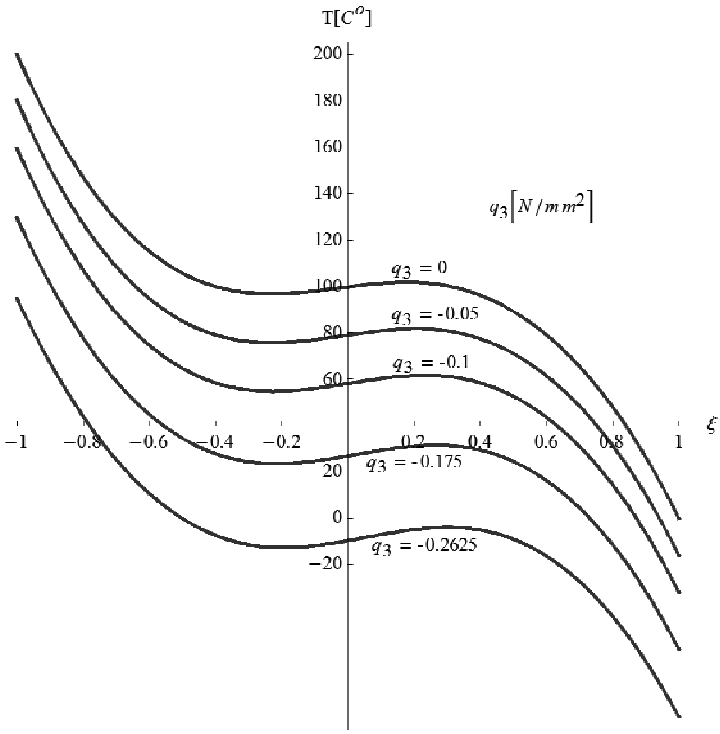

Figure 1: Stability curves for different load values q_3

In continuation let us observe the shell loaded with temperature T and force per unit area q_3 and let the outer force, equally distributed along the upper shell surface be compensated at the four opposite corners of the simply supported bimetallic shell. The reduced transversal shear forces per unit of length T_{13}^R and T_{23}^R are:

$$T_{13}^R\Big|_{x_1=a} = T_{23}^R\Big|_{x_2=a} = \lim_{\chi\to 0}\frac{-F/2}{4\chi} \tag{7}$$

where $\chi\left[mm\right]$ denotes the length of the edge at the corner of the shell, where the reduced transversal forces per unit of length T_{13}^R and T_{23}^R are exerted. The results for this example of load are in Table 2.

Table 2: The snap-through temperatures in dependence from the load q_3 in the case of a shell supported at the corners

$q_3 = \frac{F}{4a^2}\left[N / mm^2\right]$	$F = -20$ $q_3 = -0,05$	$F = -40$ $q_3 = -0,1$	$F = -70$ $q_3 = -0,175$	$F = -88$ $q_3 = -0,22$	$F = -105$ $q_3 = -0,2625$
$T_{p1}\left[C^\circ\right]$	$T_{p1} = 78,4$ $\xi_1 = 0,27$	$T_{p1} = 54,4$ $\xi_1 = 0,36$	$T_{p1} = 20,3$ $\xi_1 = 0,40$	$T_{p1} = 0,0$ $\xi_1 = 0,44$	$T_{p1} = -19,8$ $\xi_1 = 0,47$
$T_{p2}\left[C^\circ\right]$	$T_{p2} = 72,5$ $\xi_2 = -0,33$	$T_{p1} = 45,1$ $\xi_2 = -0,27$	$T_{p2} = 5,4$ $\xi_2 = -0,28$	$T_{p2} = -19,2$ $\xi_2 = -0,28$	$T_{p2} = -46,2$ $\xi_1 = -0,28$

Let us also observe the case of the simply supported shell, which at its apex, at the point $x_1 = x_2 = 0$, is loaded with a concentrated force F. For this example of load it is necessary to place in the BVP (1), (2):

$$\begin{aligned} q_3 &= \frac{F}{4\chi^2} \qquad & -\chi \le \left(x_1 \wedge x_2\right) \le \chi \\ q_3 &= 0 \qquad & -\chi > \left(x_1 \vee x_2\right) > \chi \end{aligned} \tag{8}$$

The reduced transversal shear forces per unit of length T_{13}^R and T_{23}^R have values in equation (7). Parameter h defines the region where the force F exerts and limits towards zero. The snap-through temperatures in dependence from an external force at the apex of the shell are written in Table 3, while the shape of the shell at the moment of the upper snap-through with the force $F = -105\,N$ is shown in Figure 2. In Figure 3 the local concavity of the shell due to the concentrated force at the apex of the shell is evident.

Table 3: The snap-through temperatures in dependence from an external force F in the case of a simply supported shell

$F\left[N\right]$	$F = -20$	$F = -40$	$F = -70$	$F = -105$	$F = -132$
$T_{p1}\left[C^\circ\right]$	$T_{p1} = 86,7$ $\xi_1 = 0,24$	$T_{p1} = 72,5$ $\xi_1 = 0,27$	$T_{p1} = 48,3$ $\xi_1 = 0,32$	$T_{p1} = 21,1$ $\xi_1 = 0,33$	$T_{p1} = 0,0$ $\xi_1 = 0,40$
$T_{p2}\left[C^\circ\right]$	$T_{p2} = 82,3$ $\xi_2 = -0,32$	$T_{p2} = 66,3$ $\xi_2 = -0,28$	$T_{p2} = 43,1$ $\xi_2 = -0,25$	$T_{p2} = 17,1$ $\xi_2 = -0,23$	$T_{p1} = -2,9$ $\xi_1 = -0,20$

At the end, we will treat a temperature-loaded shell where all four corners are fixed in such way allowing only rotations at the corners, while the rest of the shell is free to rotate and displace. In other words, a shell fixed so cannot expand horizontally at the corners. Instead of the normal forces N_1 and N_2 at the boundary conditions (2) we now take into account that the horizontal displacement at the corners of the shell is equal to zero:

$$w\cos\varphi\sin\psi + u\cos\psi - v\sin\varphi\sin\psi\Big|_{x_1 = x_2 = a} \cong w\,y_1' + u\Big|_{x_1 = x_2 = a} = 0 \tag{9}$$

The stability curve that shows the relation of heights ξ in dependence of the temperature T is shown for this example of load in Figure 4. When the shell is heated up to the temperature $T = 255\,C^{\circ}$ the displacements w are positive in the middle of the shell edge, and negative at the shell corners. In an overall view, the flatness of the fixed shell is decreased with heating. This fact is evident in

Figure 5 which shows the shape of the shell when it is heated to a temperature $T = 255\,C^{\circ}$. With reference to the stability curve in Figure 4, which shows that in an unstable region the relation of heights ξ increases, we can conclude that the shell does not snap-through into a convex shape

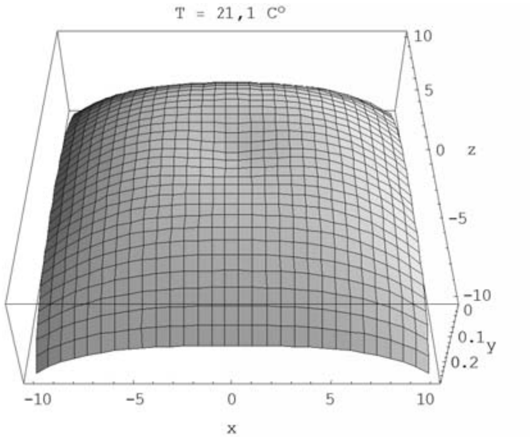

Figure 2: The geometry of the shell at the start of the upper snap-through in the case of a concentrated force $F = -105N$ acting at the apex of the shell

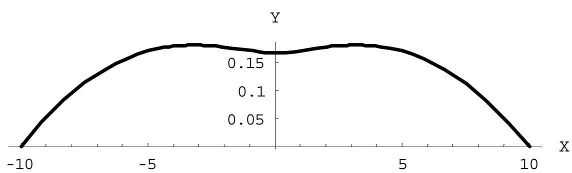

Figure 3: The occurrence of a local concavity in the case of temperature load and a mechanical force $F = -105N$ acting at the apex of the shell

Figure 6 shows the shape of a shell in the unstable equilibrium state when the shell is heated to a temperature $T = 476\,C^{\circ}$. The relation of heights ξ is at that temperature again equal to one.

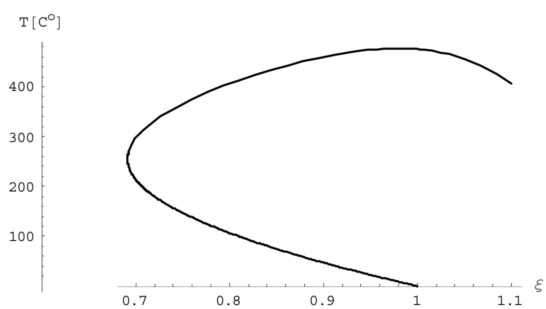

Figure 4: Stability curve for the shell of fixed corners free to rotate

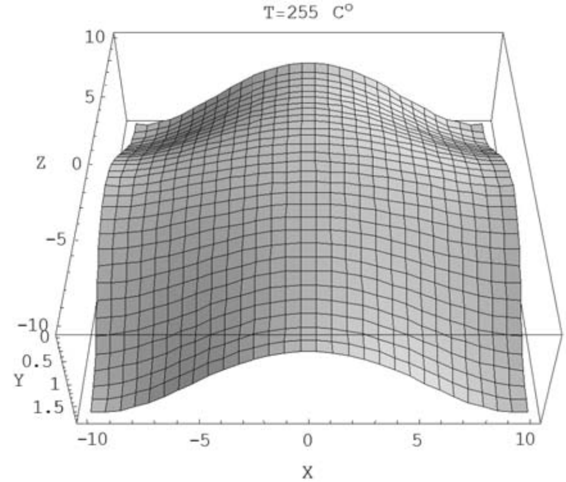

Figure 5: The geometry of a shell at the temperature $T = 255\,C^{\circ}$

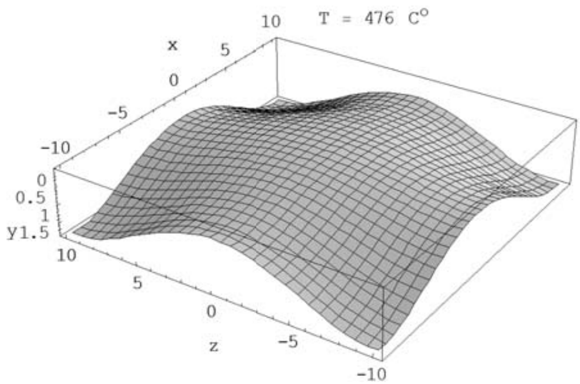

Figure 6: The geometry of the shell at temperature $T = 476\,C^{\circ}$

4. CONCLUSION

Simply supported thin-walled shallow bimetallic shells have the characteristic to snap-through into a new position at a defined temperature. The snap-through temperature T_p is dependent on the material and geometric characteristics of the shell, external mechanical loads and manner of fixation.

For shallow single layer shells with a constant coefficient of a linear temperature expansion $\alpha\left(z\right) = const.$, the relation of heights $\xi\left(T\right)$ remains constant regardless to the temperature load. With the increase in temperature T the horizontal radius does somewhat increase, while the vertical component of the displacement w at the shell edge remains the same at all times. This is why single layer shells do not have snap-through. Very shallow bimetallic shells with a small value of the parameter k of construction curves also have no snap-through. We find that a shell with the material and geometric characteristics in (4) has no temperature snap-through if its horizontal radius amounts to. Inflated or less shallow shells snap-through at higher temperatures. With an equal radius a of a bimetallic shell, the temperature of the upper snap-through T_{p1} increases with the increase of the parameter k of the construction curves.

If an external force F is exerted on the shell, snap-through will occur at a lower temperature comparing to the snap-through temperature T_{p1} of an equal shell that is loaded only with a temperature T. At which temperature the shell will snap-through is dependent, not only on the magnitude of the force F and the manner of its distribution on the shell surface, but also on the reactions at its boundaries. The snap-through temperature T_{p1} is lowest when the external force F is equalized at all four corners of the shell. With a large enough force F, the shell will snap-through without any additional temperature load.

For snap-through to occur with bimetallic shells it is necessary to ensure, apart from a high enough temperature, that the edges of the shell can freely expand. With a bimetallic free rotating shell that is fixed at the corners, displacements in a horizontal direction are not possible. Such a shell can only expand at the corners in a vertical direction due to which the increase of temperature T also increases the shell inflatedness. A shell fixed in such a manner cannot perform the function of a thermo-switch.

If an external force F is exerted on the shell, snap-through will occur at a lower temperature comparing to the snap-through temperature T_{p1} of an equal shell that is loaded only with a temperature T. At which temperature the shell will snap-through is dependent, not only on the magnitude of the force F and the manner of its distribution on the shell surface, but also on the reactions at its boundaries. The snap-through temperature T_{p1} is lowest when the external force F is equalized at all four corners of the shell. With a large enough force F, the shell will snap-through without any additional temperature load.

For snap-through to occur with bimetallic shells it is necessary to ensure, apart from a high enough temperature, that the edges of the shell can freely expand. With a bimetallic free rotating shell that is fixed at the corners, displacements in a horizontal direction are not possible. Such a shell can only expand at the corners in a vertical direction due to which the increase of temperature T also increases the shell inflatedness. A shell fixed in such a manner cannot perform the function of a thermo-switch.

5. REFERENCES

[1] Jakomin, M., Kosel, F., Kosel. T., "Buckling of a shallow rectangular bimetallic shell subjected to outer loads and temperature and supported at four opposite points." *Adv. Mech. Eng. (Online),* 767648-1-767648-17, 2009.

[2] Jakomin, M., Kosel, F., Kosel. T., "Thin double curved shallow bimetallic shell of translation in a homogenous temperature field by non-linear theory." *Thin-walled structures,* **48**(3), 243-259, 2010.

[3] Kosel, F., Jakomin, M., Batista, M., Kosel, T., "Snap-through of the system of open shallow axi-symmetric bimetallic shell by non-linear theory." *Thin-walled structures.* **44**(2), 170-183, 2006.

[4] Reddy, J. N., "Theory and Analysis of Elastic Plates." Taylor & Francis, 1999.

[5] Novozhilov, V.V., "The Theory of Thin Shells". P. Noordhoff LTD-Groningen-the Netherlands, 1959.

[6] Gould, P.L., "Analysis of plates and shells." New Jersey: Prentice Hall, 1999.

[7] Smith, G.D., "Numerical Solution of Partial Differential Equations-Finite Diffrerence methods." Oxford: Clarendon Press, Third edition 1985, Reprinted 2008.

[8] Farlow, S.J., "Partial Differential Equations for Scientists and Engineers." New York: Dover Publications.