SDSS'Rio 2010 STABILITY AND DUCTILITY OF STEEL STRUCTURES
E. Batista, P. Vellasco, L. de Lima (Eds.)
Rio de Janeiro, Brazil, September 8 - 10, 2010

A GEOMETRY BASED METHOD FOR THE STABILITY ANALYSIS OF PLATES

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Keywords: Plates, Stability, Analysis, Buckling.

***Abstract**. The objective of this paper is to introduce and investigate a new plate buckling analysis procedure based on geometry. The method is applied to a range of plate edge support condition combinations including many where results are not readily available. The results obtained by using the new procedure were compared against theoretical formulae available in the literature and by finite element analyses with good agreement. Following the verification of the new procedures the technique was extended to consider buckling of non-rectangular plates and cylindrically curved plate structures where the results were conservative but easy to use.*

1 INTRODUCTION

From the pioneering work by Bryan [1] who determined the buckling load of simply supported rectangular plates research has been carried out by many different people. These have been summarized in standard textbooks [2] – [4] and in data sheets [5] – [6]. The basic approach to determining the buckling strength of plated structures is through the solution of the linearised equations governing the transition from a flat form to a slightly buckled form. In theory, classical methods can deal with all the phenomena of flat plate stability using equilibrium, constitutive, and strain-displacement relationships. These are most easily accomplished for rectangular plates with simple boundary conditions. Numerical methods characterize the behavior of a structure at points or within regions of the structure and result in large-order systems of equations whose coefficients are numerically evaluated functions of the material, geometry, and applied-load parameters at these points or regions. As a group, these methods furnish wide latitude in the treatment of non-uniformly distributed values of the design parameters and nonlinear behavior. There are three common methods of determining the lowest buckling load of linear elastic plates, either by direct solutions of the differential equation for plates, virtual work or by use of the energy method. Bradford and Roufegarinejad [7] studied the behavior of rectangular plates with all sides clamped and with linearly varying axial edge compression. They provided a comparison of buckling analysis solutions from different investigators for square plates in pure compression and showed that small variations in the assumed models gave rise to predictions of the buckling loads varying by up to 30% (in most cases less than 5%).The objective of this paper is to develop an analysis method able to deal with all possible plate edge boundary conditions which can be used in spreadsheets for the preliminary design that is simple to apply and cost effective. The method can be applied to rectangular or parallelogram shaped plates or to rectangular curved plates.

This paper is only concerned with the determination of buckling loads for simply-supported, free and clamped conditions and not with other supports such as elastically restrained or with post-buckling considerations as the objective is to produce a new design procedure appropriate for preliminary design in aircraft structures.

2 THE GEOMETRY BASED ANALYSIS METHOD

2.1 Introduction

In formulating the plate buckling equation several parameters are needed to generalize the equation of the plate buckling problem. The parameters of the buckling equation are an applied load shape parameter [λ], a plate edge support configuration parameter [β] and a plate geometry parameter [η].

The applied load shape parameter describes the shape of the load distribution applied to the plate edges and covers plates with both axial and bi-axial loading. The plate edge support parameter describes the edge support conditions of the plate and in particular if the edge is simple, hinged or clamped. The plate geometry parameter is based on the aspect ratio of the plate geometry. Equation (1) presents the plate buckling equation according to the Geometry Based Analysis Method (GBAM)

$$\sigma_{cr} = [\sigma_{rel}][\beta][\lambda][\eta] \tag{1}$$

where σ_{cr} is the critical buckling stress and σ_{rel} is a plate relative buckling stress parameter.

2.2 Determination of the plate relative buckling stress σ_{rel}

The relative buckling stress in the loading direction is $\sigma_{rel,x}$ and in the transverse direction is $\sigma_{rel,y}$. These stresses are calculated from the Euler buckling load of a simply-supported column which is given by:

$$F_{cr} = \left(\pi^2\right)\left(\frac{EI_y}{a^2}\right) \tag{2}$$

The Euler equation deals with forces, whilst a plate analysis deals with stresses. Therefore equation (2) has to be converted into a stress problem. For a plate, which has length a in loading x-direction and width b in the transverse y-direction and constant thickness t the plate Euler buckling stresses σ_{Euler} is determined by dividing both sides of equation (2) by the plate cross section area of the loaded side (bt).

i.e.

$$\sigma_{Euler} = \frac{F_{cr}}{bt} = \left(\frac{\pi^2}{bt}\right)\left(\frac{EI_y}{a^2}\right) \tag{3}$$

The plate relative buckling stresses $\sigma_{rel,x}$ and $\sigma_{rel,y}$ are determined directly from equation (3).

$$\sigma_{rel,x} = \left(\frac{\pi^2}{bt}\right)\left(\frac{EI_y}{a^2}\right) \tag{4}$$

$$\sigma_{rel,y} = \left(\frac{\pi^2}{bt}\right)\left(\frac{EI_x}{b^2}\right) \tag{5}$$

where I_x and I_y are the respective second moments of area of the plate about centroidal axes in the plate.

2.3 Determination of the plate geometry parameter [η]

The geometry based analysis method classifies plates into two sorts: short plates and long plates. This classification is derived from the k values curve of uni-axially loaded plates simply supported on all edges according to classical buckling theory. Figure 1 shows the buckling coefficients (k) according to the aspect ratio α (α is the ratio of the length of the plate in the loaded direction divided by the width of the plate) and the number of half waves or buckles (m) on the plate in the longitudinal loaded direction. The curve shows that a plate with only one buckle, $m = 1$, intersects the curve of a plate with two buckles, $m = 2$, at the point $\alpha=1.4$. For the geometry based analysis method we consider the intersection point as the separation point between short and long plates.

The plate geometry coefficient is assumed to be linear up to an aspect ratio of 1.4 and then afterwards to mirror the single buckle value up to an aspect ratio of α = 2. Above this value it is assumed to be always equal to 1.800 as the intersection of multiple modes means that buckled modes above this value are approximately the same as can be seen in Figure 1. The values are given in Table 1.

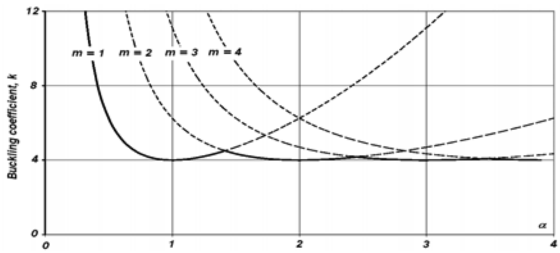

Figure 1: Buckling coefficients, *k*, for a simply supported plate

Table 1: Values of η for m = 1 and m = 2 buckles

m	1	1	1	1	1	1	1	1	1	1/2	2	2	2	2	2
α	0.000	0.250	0.500	0.625	0.750	0.875	1.000	1.125	1.250	1.400	1.550	1.675	1.800	1.925	2.000
η	1.000	1.250	1.500	1.625	1.750	1.875	2.000	2.125	2.250	2.400	2.225	2.125	2.000	1.875	1.800

Below an aspect ratio α of 1 the buckling load decreases as the aspect ratio α is increased, Above this limiting value of α the critical stress changes marginally as can be seen in Figure 1.

2.3 Determination of the plate edge boundary terms

Plates are unlike columns, plates have not only end boundary conditions like columns but also lateral boundary conditions. In other words, column buckling is resisted by only one bending stiffness (the smallest bending stiffness) whilst plate buckling is resisted by the bending stiffness of the plate in both longitudinal and lateral directions. Figure 2 shows the two terms β_x in the loading direction and β_y in lateral direction, which are linked together by the relative plate buckling stress σ_{rel}.

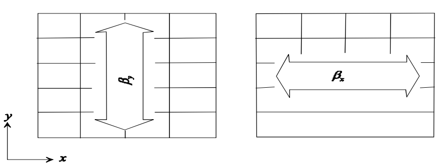

Figure 2: Edge boundary terms β_x and β_y

The standard four Euler column cases are the free-clamped, case *I*, simple-simple, case *II*, simple-clamped, case *III* and clamped-clamped, case *IV*. The critical compression force F_{cr} for these columns is given by the standard Euler formulae. For example in case *III*:

$$F_{cr} = 2.04\pi^2 \frac{EI_y}{a^2} \tag{6}$$

Defining case *II* as the "Basic" case we relate the other three cases to determine the values of β^I_{Kx}, β^I_y, β^{III}_x, β^{III}_y, β^{IV}_x and β^{IV}_{Ky} where the case number is written as a superscript. The value of β^{II}_x equals 1 whilst the value of β_y^{II} equals 0 since there is no lateral support. As the buckling coefficient for the clamped-free case is 0.25 we get $\beta_x^{I} = 0.25$ and $\beta_y^{I} = 0$. In similar manners $\beta_x^{III} = 2.04$, $\beta_y^{III} = 0.00$, $\beta_x^{IV} = 4.00$ and $\beta_y^{IV} = 0.00$.

Two virtual buckling load cases – Case *V*, free-free and case *VI*, free-simply supported are now defined. Obviously $\beta_x^{V} = 0.00$ and $\beta_y^{V} = 0.00$. The 6 cases are plotted on a straight line where the assumption is made that case *VI* is between case *V* and case *I*. The resulting plot is shown in Figure 3.

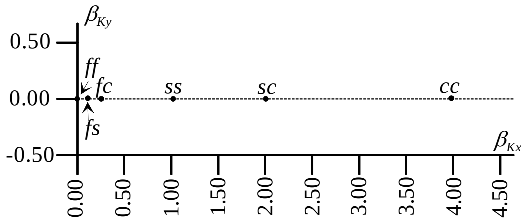

Figure 3: Stiffness terms for β_x and β_y for the Euler cases and the virtual cases

In order to determine β_x^{VI} for the case free-simple the following correspondences are used: cases *I*, *V* and *VI* all have one free edge; Cases *I*, *III* and *IV* have one clamped edge; both sets are linked by case *I*. The value of case *VI* is determined by proportion

$$\frac{\beta_x^{VI} - \beta_x^{V}}{\beta_x^{I} - \beta_x^{V}} = \frac{\beta_x^{III} - \beta_x^{I}}{\beta_x^{IV} - \beta_x^{I}} \tag{7}$$

Solving for β_x^{VI} and substituting

$$\beta_x^{VI} = \left(\frac{\beta_x^{III} - \beta_x^{I}}{\beta_x^{IV} - \beta_x^{I}}\right)\left(\beta_x^{I} - \beta_x^{V}\right) + \beta_x^{V} = \left(\frac{2.04 - 0.25}{4.00 - 0.25}\right)(0.25 - 0.00) = 0.119 \tag{8}$$

In order to extend this column analogy into plates the edge ordering shown in Figure 4 is used:

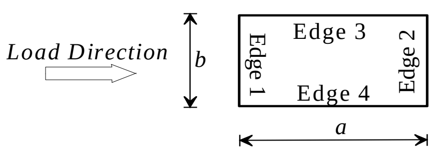

Figure 4: Edge configuration order

18 possible edge configurations can be identified as shown in Figure 5. They are sorted into three groups – Group 1 with loaded edges simply supported, Group 2 with one loaded edge simply-supported and one clamped and Group 3 with both loaded edges clamped.

Lateral support condition

	ff	*fs*	*fc*	*ss*	*sc*	*cc*	Loaded edges support condition
Group 1	*ssff* 10	*ssfs* 18	*ssfc* 17	*ssss* 01	*sssc* 05	*sscc* 07	*ss*
Group 2	*scff* 12	*csfs* 13	*scfc* 16	*csss* 06	*scsc* 03	*sccc* 08	*sc*
Group 3	*ccff* 11	*ccfs* 15	*ccfc* 14	*ccss* 04	*ccsc* 09	*cccc* 02	*cc*

Figure 5: Edge support conditions

The calculation approach is to relate the buckling of the plates to the Euler column buckling cases. The unknown values of β_x and β_y are estimated using simple interpolation.

Considering the plates shown in Figure 5 let $\Delta\beta^{01-05}$ be the increment from the basic case, plate 01, to plate 05. The increment $\Delta\beta^{01-05}$ is calculated from the Euler column cases, case *I* and case *VI* as follows:

$$\Delta\beta^{01-05} = 0.119 + 0.25 = 0.369 \tag{9}$$

Let $\Delta\beta^{05-07}$ be the increment from plate 05 to plate 07. The increment $\Delta\beta^{05-07}$ is twice the previous increment as the difference in buckling factors in going from the propped cantilever to the fully fixed case is approximately two. Therefore $\Delta\beta^{05-07}$ = 2(0.369) = 0.738.

In a similar manner, the increment $\Delta\beta^{05-03}$ is also twice the increment $\Delta\beta^{01-05}$ and hence is also 0.738.

The increment $\Delta\beta^{03-02}$ equals twice the previous increment $\Delta\beta^{01-03}$ and is therefore 2(0.738) =1.476.

Once the increments are known, we add them to the values of β_{Kx} and β_{Ky} of the plates starting from plate 01, i.e. β_x^{1} and β_y^{1} which are known a priori as equal to 1.000.

Hence we can establish, for example

$$\beta_x^{03} = \beta_x^{07} = 1.000 + 0.738 = 1.738 \tag{10}$$

$$\beta_x^{02} = 1.738 + 1.476 = 3.214 \tag{11}$$

Using similar principles we can fill in the remaining values of β_x and β_y for cases 01-09.

For plates with lateral edges free the values of β_x and β_y have to be modified. The values of β_x and β_y are determined as before by calculating the decrements. For example,

$$\Delta\beta^{01-17} = 0.5 \bullet 0.369 = 0.185 \tag{12}$$

$$\Delta\beta^{17-18} = 0.5 \bullet 0.185 = 0.093 \tag{13}$$

Hence

$$\beta_{Kx}^{17} = 1.000 - \Delta\beta^{01-11} = 1.000 - 0.185 = 0.815 \tag{14}$$

$$\beta_{Kx}^{18} = 1.000 - \Delta\beta^{17-18} = 1.000 - 0.093 = 0.907 \tag{15}$$

From Equations (14) and (15) it can be seen that simply-supported edges have a reduction factor of 0.815 and clamped edges a reduction factor of 0.907. These factors are used in determining the reduction factors for the remaining combinations. The resulting sets of β_x and β_y are given in Table 2.

Table 2: Values of β_x and β_y for all 18 plate combinations

Case	01	02	03	04	05	06	07	08	09
Edge	*ssss*	*cccc*	*scsc*	*ccss*	*sssc*	*csss*	*sscc*	*sccc*	*ccsc*
β_x	1.000	3.214	1.738	3.214	1.000	1.738	1.000	1.738	3.214
β_y	1.000	1.738	1.369	1.000	1.368	1.000	1.738	1.738	1.369
Case	10	11	12	13	14	15	16	17	18
Edge	*ssff*	*ccff*	*scff*	*csfs*	*ccfc*	*ccfs*	*scfc*	*ssfc*	*ssfs*
β_x	0.800	2.893	1.564	1.564	2.893	2.893	1.564	0.800	0.800
β_y	0.000	0.000	0.000	0.095	0.2225	0.095	0.225	0.225	0.095

3 EXAMPLES

In this section a few examples are presented treating different edge configurations of the plates. All examples deal with rectangular aluminium plates with thickness t = 1.0mm. The aspect ratio α varies from α = 0.3 to α = 2.0. Young's Modulus of elasticity E = 70GPa and Poisson's ratio ν = 0.3. The results of the classical theory were estimated using Bulson [4] or the software DLUBAL/RSTAB, whilst the FEM eigenvalue buckling calculations were computed using the software MSC/NASTRAN. The results from the method proposed in this analysis are called GBAM. The applied edge compression stress is axially uniform. The results are presented in term of the buckling constant k, where all the results of calculated critical stresses are divided by a reference stress σ_E given by:

$$\sigma_E = \frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{b}\right)^2 \tag{16}$$

Four examples of plates under uniform compression are given in Figures 6-9.

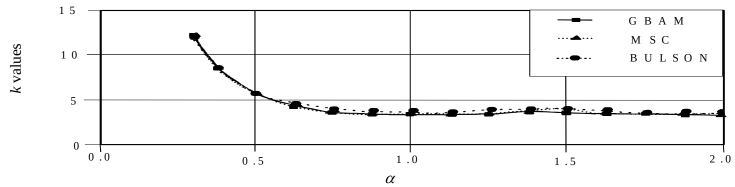

Figure 6: Buckling coefficients for a plate simply-supported on all sides (Case 01 *ssss*)

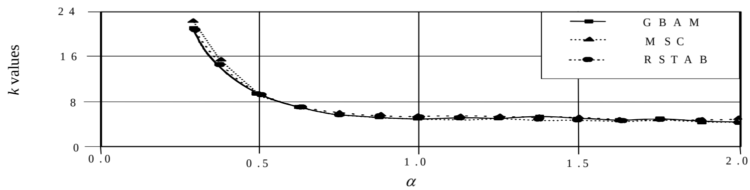

Figure 7: Buckling coefficients for a plate case 06 (*csss*)

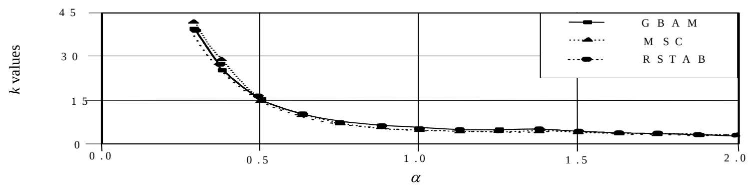

Figure 8: Buckling coefficients for a plate case 09 (*ccsc*)

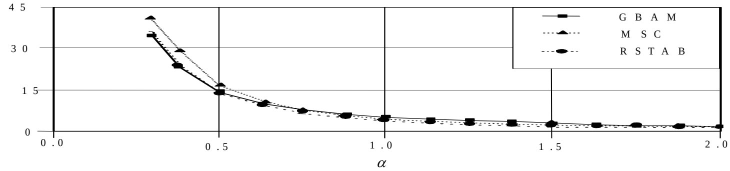

Figure 9: Buckling coefficients for a plate case 14 (*ccfsc*)

It is noticeable that in all cases the GBAM approach gives results as accurate as those of finite element analyses.

4 EVOLUTIVE PLATES AND CURVED PLATES

Equation (1) describes only the case of rectangular plates. To extend the geometry based plate stability analysis method to include also non-rectangular plates needs modification of the plate relative buckling stress parameter σ_{rel}. Figure 10 shows the the evolutive plate used in this analysis.

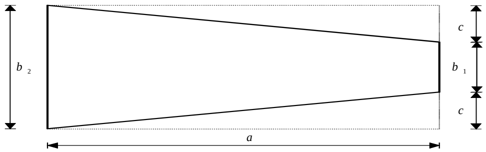

Figure 10: Dimensions of evolutive plates

Consider a unit square plate which has $\alpha = a/b_2 = 1.0$ and $a = 1.0$. The change in the longitudinal plate relative buckling stress $\sigma_{rel,x}$ of the plate is related to the change of the aspect ratio α from the long edge b_2 to the short edge b_1 of the evolutive plate. The modification parameter [δ_x] is written as:

$$\delta_x = \alpha_1 \bullet \alpha_2 = \frac{a}{b_1} \bullet \frac{a}{b_2} = \frac{1}{b_1} \bullet \frac{1}{1} = \frac{1}{b_1} \tag{16}$$

The change of the plate relative buckling stress $\sigma_{rel,y}$ in the lateral direction is governed by the lateral change in area of the square plate and the evolutive plates, i.e. the plate with evolutive edges is converted into an equivalent square plate. The difference in area, ΔA, is equated to a square plate and the edge length is added to the width of the evolutive plate. Accordingly the removed area equals:

$$\Delta A = 2\left(\frac{1}{2} c \bullet 1\right) \tag{17}$$

The change in edge length, Δb equals

$$\Delta b = \sqrt{c} \tag{18}$$

Hence

$$b_2 = 1 + \sqrt{c} \tag{19}$$

δ_y is taken equal to the new value of b_2.

Hence, for example, if $c = 0.3$ then $b_1/ b_2 = 0.4$, $\delta_x = 1.0/0.4 = 2.5$ and $\delta_y = 1 + (0.3)^{0.5} = 1.55$. Using mean stresses similar equations can be constructed for plates with different stresses at each end. Figure 11 shows the results of a typical analysis with different b_1/ b_2 ratios for simply-supported plates. The results of the analysis are compared against those tabulated in the German handbook [9]. It is noticeable that the largest discrepancies occur for the case $b_1/ b_2 = 1.0$ whereas the agreement between GBAM and finite element procedures is excellent as can be seen in Figure 6.

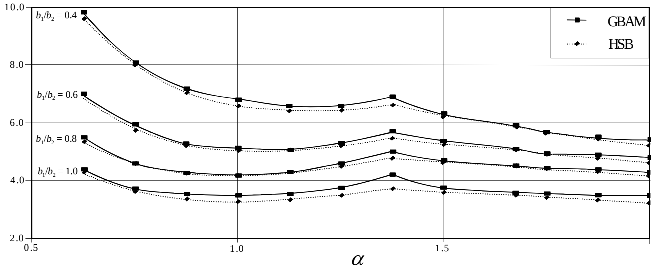

Figure 11: Buckling coefficients for evolutive plates

In order to extend the procedure to slightly curved plates (plates bent cylindrically with radius r and the outer edges subtending an angle 2ϕ radians at the centre) the moments of inertia I_y in the stiffness matrix have to be modified. In these calculations the plan breadth, b, is used. The curvature will increase the stiffness of the plate, - since the centre of gravity (CG) of the curved plate is displaced by a distance d_{CG} from the centre of gravity of the flat plate. The moment of inertia I_y is then calculated:

$$I_y = \frac{bt^3}{12} + bt\left(d_{CG}\right)^2 \tag{20}$$

where

$$d_{CG} = \left(r\cos\phi + \frac{t}{2}\right) - \frac{d_1 A_1 - d_2 A_2}{A_2 - A_1} \tag{21}$$

and $d_1 = \frac{2(r+t)\sin\phi}{3\phi}$, $d_2 = \frac{2r\sin\phi}{3\phi}$, $A_1 = (r+t)^2\phi$, $A_2 = r^2\phi$

The maximum curvature that the procedure can be applied to is shown in reference [8] to be

$$100b/(\alpha r) < 1 \tag{22}$$

For example, consider a curved aluminum plate, thickness 2.0mm, width 50.0 mm, α = 0.3 and radius r = 20 m. For this plate equation (22) yields the ratio to be 0.83 and using equation (21) d_{CG} = 0.04 mm. Hence using equation (20) I_y = 33.51 mm^4 and I_x = 10.00 mm^4. From equations (4) and (5) $\sigma_{rel,x}$ = 1028.85 N/mm^2 and $\sigma_{rel,y}$ = 27.63 N/mm^2. The total σ_{rel} = 1056.48 N/mm^2. Finally using equation (1) the critical buckling stress is 1373 N/mm^2. This compares with the stress obtained using reference [9] which is 1368 N/mm^2.

Similar accuracies are obtained when the method is applied to other curved plates.

5 CONCLUSIONS

This paper has presented the development of a new procedure based on geometry for the design and analysis of plate buckling. The results of the procedure have been compared against both finite element analyses and classical analyses presented in the literature and have been shown to be accurate when applied to rectangular, evolutive and slightly curved plates.

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